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Comparing the finite and infinite limits of sequences and functions: A mathematical and phenomenological analysis and its implications in Spanish textbooks

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The purpose of this article is to conduct a mathematical and phenomenological comparison of three concepts: (1) the finite limit of a function at a point, (2) the finite limit of a sequence, and (3) the infinite limit of a sequence. Additionally, we aim to analyse the presence of these concepts in Spanish textbooks. The methodology employed is exploratory and descriptive. Our mathematical comparison revealed differences in several areas, including the dependence between variables, the involved infinite processes, the types of infinity, the dimensioning for each variable, and the intuition of continuity in the interval. Our phenomenological comparison found a correspondence between phenomena using a formal approach, but differences in phenomena when using an intuitive approach. Finally, our analysis of textbooks revealed that all three limits are most commonly presented in the verbal representation system and definition format.

Contribution: This study has contributed to the teaching and learning of the notion of limit.

Keywords: Finite limit; infinite limit; sequence; function; mathematics comparison; phenomenology; textbooks; intuitive approach.

Introduction

The concept of limit originated in ancient Greece as a means to validate results derived from approximations of geometric figures using the method of exhaustion. Its utilisation persisted until the 17th century when Newton expanded upon the Greek interpretation. He reimagined the approximation of the limit by associating variable quantities with moving physical bodies (Camacho & Aguirre, 2001). During the latter part of the 17th century and throughout the 18th century, mathematicians delved into infinite processes, leading to a distinct separation of calculus from geometry (Blázquez et al., 2008). Throughout this period, calculus underwent historical evolution and encountered challenges, including the absence of a rigorous foundation and debates over the interpretation of infinitesimals, some of which persist to this day.

In the literature review conducted separately by Claros (2010), Sánchez (2012), and Arnal-Palacián (2019) within their respective doctoral theses, a notable observation emerged: authors delving into the study of limits often failed to distinguish between the limit of sequences and functions. Additionally, they did not differentiate between finite and infinite limits when addressing this concept. The overarching thesis posited by these scholars highlights that resolving some of the difficulties associated with limits requires a differentiated approach to each type of limit. These challenges, previously identified by various authors (Sierra et al., 1999; Blázquez & Ortega, 2000) and endorsed by the aforementioned researchers, have steered our research team toward a detailed examination of each limit type.

It is essential to have knowledge of the difficulties, obstacles, and errors in the concept of limit. Some research has focused on this mathematical notion with students as the main focus (Arnal-Palacián et al., 2022; Claros, 2010; Douglas, 2018; Kidron, 2011; Jirotková & Littler, 2003; Jutter, 2006; Morales et al., 2013; Valls et al., 2011), while others have studied pre-service and in-service teachers (Arnal-Palacián & Claros-Mellado, 2022; Arnal-Palacián et al., 2022; Kattou et al., 2009; Lestón, 2012; Movshovitz & Hadass, 1990; Sánchez, 2012). Claros (2010) categorised challenges related to the concept of limit into three distinct types: difficulties concerning mathematical analysis, issues specific to the concept of limit, and challenges associated with the language used to describe the limit. Obstacles have been identified within the concept itself, within teaching methodologies, and within students' cognitive structures (Irazoqui & Medina, 2013). Moreover, the colloquial use of the term 'limit' often conveys an interpretation of an insurmountable barrier or the conclusion of a process (Cornu, 1991). Additionally, common phrases such as 'tend', 'approach,' or 'approaching' pose difficulties (Tall & Vinner, 1981).

One approach to mitigate these challenges involves employing diverse representations (Vrancken et al., 2006). Despite the prevalent trend in calculus education favouring an algorithmic and algebraic approach, alternative studies suggest the importance of investigating each limit individually (Morales et al., 2013).

There are many didactic proposals for approaching the teaching of the limit, some focused on students and others on teachers. All try to help both to overcome the difficulties surrounding the limit (Rojas, 2015).

Precisely in this same perception, studies on the phenomenology of the finite limit of a sequence (Claros, 2010), the finite limit of a function at a point (Sánchez, 2012) and the infinite limit of a sequence (Arnal-Palacián, 2019) were carried out.

Each of these authors conducted comprehensive mathematical and phenomenological studies of the aforementioned notion, aiming to alleviate the historical (Boyer, 1992) and current challenges associated with the concept of limit (Fernández et al., 2018). While we acknowledge that there remain types of function limits to address, such as finite limits at infinity and infinite limits at infinity, it is essential to prioritise an examination of these previous types. Subsequently, a comparative analysis—both mathematically and phenomenologically—between the studied limit types becomes necessary. The rationale behind this lies in our belief that such a comparison will undoubtedly reinforce the authors' thesis advocating for the differentiated approach to each type of limit.

Phenomenology, a philosophical discipline that began its development in the 20th century, has also found discussion within mathematics education, particularly as examined by Freudenthal (1983). Notably, Freudenthal's interpretation of phenomenology differs from those of Hegel, Husserl, or Heidegger. However, Freudenthal does not extensively elucidate his divergences from these philosophers; instead, he confines his explanation to affirming that 'noumenon' constitutes the object of thought, while 'phenomenon' represents something that is experiential.

This sense of phenomenology is interpreted as the component of his didactic analysis in which he starts from the contrast between the terms noumenon, *noumenon*, and phenomenon, *phainomenon*. This philosophical reflection is based on the contrast between the objects constructed in concepts, which are called objects of thought and which will be called 'noumena', and the situations that these mathematical objects organise, when one has acquired experience, which will be the 'phenomena'. In every exposition of a concept, Freudenthal begins with an example and continues with the characterisation. Gómez (2007) and Gómez and Cañadas (2011) perceive phenomenology as an integral aspect of defining a mathematical concept, stemming from a functional viewpoint within the curriculum. According to this perspective, the usage and application of a concept encompass the phenomena it underpins.

We will adopt this stance to foster the advancement of the limit concept, specifically exploring the finite and infinite limits of a sequence, along with the finite limit of a function at a point. Our aim is to create an effective framework to address and mitigate the comprehension issues encountered by students when engaging with this concept.

Freudenthal (1983) does not dedicate a specific section to the study of sequences, but does so for functions. We can understand that this is the result, as was the case with other prestigious mathematicians such as Spivak (1994, p. 445), of the fact that he considers that 'an infinite sequence of real numbers is a function whose domain is N'.

Claros et al. (2007) conducted an in-depth investigation characterising phenomena related to the finite limit of a sequence and the finite limit of a function at a point. This study also involved a mathematical and phenomenological comparison of these phenomena. Expanding upon this groundwork, the present study endeavours to augment these insights by incorporating the notion of the infinite limit of a sequence. As such, the following research objectives have been formulated:

- To mathematically compare the finite and infinite limit of sequences and the finite limit of a function at a point.
- To perform a phenomenological comparison of the finite and infinite limits of sequences and the finite limit of a function at a point, as per the definition by Freudenthal (1983).
- To compare the representation of these phenomena concerning each limit in a selection of Spanish textbooks.

Method

The present study is exploratory in nature and aims primarily to describe the phenomena under investigation (Elliott & Timulak, 2005). Employing a qualitative approach, the study encompasses three complementary analyses.

The initial analysis focuses on a mathematical comparison between the finite and infinite limits of sequences and the finite limit of a function at a point. This examination delves into mathematical notions such as dependence, infinite processes, types of infinity, boundedness, and intuition regarding interval continuity.

The second analysis involves a phenomenological comparison, adhering to Freudenthal's conception, and centres on characterising the observed phenomena. It initiates by scrutinising diverse definitions of the notions of the finite limit of a function at a point, finite limit of a sequence, and infinite limit of a sequence. This investigation reveals that these notions can be approached either intuitively or formally.

Intuitive phenomena manifest themselves when contemplating the dynamic aspect of limits in a non-rigorous manner. However, upon conducting a rigorous analysis of these notions, formal definitions emerge. Among the various definitions explored and following consultations with experts, we selected the formal definitions of the notions that were deemed most appropriate by experts; Claros (2010), Sánchez (2012), and Arnal-Palacián (2019). These formal definitions are presented in Table 1.

It is precisely on the basis of the above definitions, the intuitive and formal approaches, and the phenomenology given by Freudenthal, that we consider the phenomena characterised in previous studies (Arnal-Palacián, 2019; Claros, 2010; Sánchez, 2012) for the phenomenological comparison (see Table 2).

TABLE 1: Selected definitions from previous studies.

Type of limit	Selected definition
Finite limit of a function at a point	The function f approaches the limit L near a means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < \lvert x - a \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$. (Spivak, 1994, p. 96)
Finite limit of a sequence	A sequence $\{a_n\}$ converges to L (in symbols $\lim_{n\to\infty} a_n = L$) if every $\varepsilon > 0$ there is a natural number N such that, for all numbers n, if $n \geq N$, then $\lvert a_n - L \rvert < \varepsilon$. (Spivak, 1994, p. 446).
Infinite limit of a sequence	Let K be an ordered body, and $\{a_n\}$ a sequence of elements of K. The sequence $\{a_n\}$ has by limit 'plus infinity', if for each element H of K, there exists a natural number v, such that $a_n > H$, for every $n \geq v$. (Linés, 1983, p. 29).

Note: Please see the full reference list of the article, Arnal-Palacián, M., Claros-Mellado, FJ. Sánchez-Compaña, MT. (2024). Comparing the finite and infinite limits of sequences and functions: A mathematical and phenomenological analysis and its implications in Spanish textbooks. *Pythagoras, 45*(1), a774. https://doi.org/10.4102/pythagoras.v45i1.774, for more information.

Figure 1, Figure 2, and Figure 3 show some examples to illustrate these definitions.

The third and final comparative analysis aims to investigate the presence of each of the examined limits and the manifestation of associated phenomena within a sample of Spanish textbooks. An exploratory empirical study was conducted using a purposive sample of textbooks available to the researchers, published between 1936 and 2021, sourced from various publishing houses. The sample specifically included textbooks that featured any of the characterised phenomena related to the studied limits. Textbooks that did not address the notion of limits or failed to identify any of the characterised phenomena were excluded from the sample. Consequently, the sample size varied for each limit: $N = 30$ for the finite limit of a sequence, $N = 28$ for the finite limit of a

Left of x_0		Right of de x_0
X	*F(x)*	*X*
1	3	
	5	3
1.5	3.5	
	4.5	2.5
1.8	3.8	
	4.2	2.2
1.9	3.9	
	4.1	2.1
1.95	3.95	
	4.05	2.05
1.99	3.99	
	4.01	2.01
1.999	3.999	
	4.00	2.00
1.9999	3.9999	
	4.0001	2.0001

Source: Adapted from Vizmanos, J.R., Anzola, M., & Primo, A. (1981). *Funciones-2 matemáticas 2° B.U.P.* Teoria y Problemas. Ed. SM

FIGURE 1: Intuitive double approximation (i.d.a.) in tabular representation system.

TABLE 2: Characterisation of phenomena.

Type of limit	Phenomena
Finite limit of a function at a point (Sánchez, 2012)	*Intuitive double approximation* (*i.d.a.*). Given *k* pairs of values of a real function *f* of real variable ($x1, f(x1)$), ($x2, f(x2)$), ... , ($xk, f(xk)$), we identify the intuitive double approximation as the phenomenon that occurs when, in a related way, the values $x1, x2, \ldots, xk$ and their respective images $f(x1)$, $f(x2), \ldots, f(xk)$ seem to approach different fixed values. The learner believes that there are two approaches, that of the sequence of values of the independent variable towards a value and that of the sequence of values of the dependent variable towards the limit; he is aware, or not, of the connection that the function *f* establishes between both sequences. *Feedback or One Way and Return in Functions* (*o.w.r.f.*). The analysis of the definition gives rise to the observation of two closely related processes: • The first process, called 'one way' corresponds to the following fragment of the definition appears in the expression 'for every $\varepsilon > 0$ there is some $\delta > 0$'. • The second process, called 'return', corresponds to the fragment of the definition "if $0 < \lvert x - a \rvert < \delta$, then $\lvert f(x) - L \rvert < \varepsilon$".
Finite limit of a sequence (Arnal-Palacián, 2019)	*Intuitive Simple Approximation* (*i.s.a.*). Given *k* ordered terms of a sequence, usually consecutive (1, *a*1), (2, *a*2) ... (*k*, *ak*) we characterise the Intuitive Simple Approximation as the phenomenon observed when inspecting the sequence of values *a*1, *a*2, ... , *ak* as they 'appear to approach' another fixed value. *Feedback or One Way and Return in Sequence* (*o.w.r.s.*). Analysis of the definition of the finite limit of a sequence gives rise to the observation of processes: • The first process, called 'one way', occurs when the definition contains the expression 'if every $\varepsilon > 0$ there is a natural number N' • The second process, called 'return', corresponds to the fragment 'if $n \geq N$ then $\lvert a_n - L \rvert < \varepsilon$'
Infinite limit of a sequence (Arnal-Palacián, 2019)	*Unlimited Intuitive Growth* (*u.i.-g.*). An increasing sequence fulfils the idea that the values of the sequence become larger and larger. If $n > m$, then $s(n) > s(m)$ ($s(n)$ general term of the sequence). By checking this for several values, we intuitively deduce that the sequence is increasing. *Unlimited Intuitive Decrease* (*u-i.-d.*). A decreasing sequence fulfils the idea that the values of the sequence become smaller and smaller, small being understood as those negative numbers whose absolute value is greater and greater. If $n > m$, then $s(n) < s(m)$. *Feedback or One Way and Returned in Sequences with an Infinite Limit* (*o.w.r.s.i.*). Analysis of the definition of the infinite limit of a sequence gives rise to the observation of processes: • The first process, called 'one way', corresponds to the fragment: 'if for each element H of K, there exists a natural number v'. • The second process, called 'return', corresponds to the fragment 'such that $a_n > H$, for every $n \geq v$'.

Note: Please see the full reference list of the article, Arnal-Palacián, M., Claros-Mellado, FJ. Sánchez-Compaña, MT. (2024). Comparing the finite and infinite limits of sequences and functions: A mathematical and phenomenological analysis and its implications in Spanish textbooks. *Pythagoras, 45*(1), a774. https://doi.org/10.4102/pythagoras.v45i1.774, for more information.

Por ejemplo: la sucesión $\frac{2n+1}{3n+5}$ tiene por limite $\frac{2}{3}$ pues la diferencia

$$\left| \frac{2n+1}{3n+5} - \frac{2}{3} \right| < \varepsilon, \text{ siempre que } n > n' = \frac{7-15\varepsilon}{9\varepsilon} \text{ y culquiere}$$

que sea $\varepsilon > 0$.

Translation

For example: the sequence $\frac{2n+1}{3n+5}$ has $\frac{2}{3}$ as its limit, so the difference

$$\left| \frac{2n+1}{3n+5} - \frac{2}{3} \right| < \varepsilon, \text{ must always } n > n' = \frac{7-15\varepsilon}{9\varepsilon} \text{ and any } \varepsilon > 0.$$

Source: Martínez-Losada, A., Hernández Aina, F., & Lorenzo Miranda, F. (1976). *Matemáticas 2° BUP*. Editorial Tecnibán

FIGURE 2: One Way and Return in Sequence (o.w.r.s.) in symbolic representation system.

Comprobamos que:

$$\lim_{x\to\infty}(n^2-1) = +\infty$$

Dado un valor de k muy grande, por ejemplo k = 10 000. buscamos h tal que para cualquier $n > h$ se cumple que $a_n > 10\,000$

Si $h = 50 \xrightarrow{n>h} n = 51$

$a_{51} = 51^2 - 1 \ngtr 10\,000$

Si $h = 50 \xrightarrow{n>h} n = 101$

$a_{101} = 101^2 - 1 > 10\,000$

Obtenemos el mismo resultado para $n = 102, 103, \ldots$ Es decir, para cualquier $n > h$

Translation:

We check that:

$$\lim_{x\to\infty}(n^2-1) = +\infty$$

Given a very large value of k, for example, k = 10 000 we are looking for an h such that for any $n > h$ it is satisfied that $a_n > 10\,000$.

Si $h = 50 \longrightarrow$ if $n > h,\ n = 101$

$a_{51} = 51^2 - 1 \ngtr 10\,000$

Si $h = 50 \longrightarrow$ if $si\ n > h,\ n = 101$

$a_{101} = 101^2 - 1 > 10\,000$

We obtain the same result for $n = 102, 103, \ldots..$ That is, for any $n > h$

Source: Escoredo, A., Gómez, M.D., Lorenzo, J., Machín, P., Pérez, C., Del Río, J., & Sánchez, D. (2009). *2° Bachillerato. Matemáticas II*. Ed. Santillana

FIGURE 3: One Way and Returned in Sequences with an Infinite Limit (o.w.r.s.i.) in symbolic and verbal representation system.

function at a point, and $N = 35$ for the infinite limit of a sequence.

In addition to a qualitative study, a frequency analysis will be conducted. Due to varying sample sizes, we will divide the absolute frequencies obtained for the identified phenomena within a specific limit by the total number of books analysed for that limit. This calculation will yield what we refer to as 'the average number of phenomena in each book'.

The qualitative analysis considers the following variables: approaches, systems of representation, format, and the type of limit. These variables have previously enabled us to analyse textbooks on the limit up to 2005 (Claros et al., 2016) and compare the occurrence of the infinite limit of a sequence across different Spanish educational laws (Arnal-Palacián et al., 2020). Including the type of limit as a variable will facilitate a comprehensive comparison between them. The four variables and their corresponding categories are detailed in Table 3.

Results and discussion

The subsequent sections present the outcomes derived from the mathematical comparison, the phenomenological comparison, and the examination of the finite and infinite limits of a sequence, along with the finite limit of a function at a point, as observed in textbooks.

TABLE 3: Variables and categories.

Variables	Categories
Phenomena	• i.d.a. and o.w.r.f. for the finite limit of a function at a point (Sánchez, 2012), • i.s.a. and o.w.r.s. for the finite limit of a sequence (Claros, 2010), • u.i.-g., u.i.-d and o.w.r.s.i. for the infinite limit of a sequence (Arnal-Palacián, 2019)
Approaches	• Intuitive • Formal
Representation systems (Janvier, 1987)	• Verbal • Tabular • Graphic • Symbolic
Formats	• Definition • Example

Note: Please see the full reference list of the article, Arnal-Palacián, M., Claros-Mellado, FJ. Sánchez-Compaña, MT. (2024). Comparing the finite and infinite limits of sequences and functions: A mathematical and phenomenological analysis and its implications in Spanish textbooks. *Pythagoras, 45*(1), a774. https://doi.org/10.4102/pythagoras.v45i1.774, for more information.

Mathematical results

The notion of limit has two dependencies. The first dependence is found in the very definition of sequence or function, for which each natural number (sequence) or real number (function) corresponds to a real number, and in which the notion of limit does not intervene. In the second, the real number is related to the natural number (sequence) or real number (function) that it occupies. The first dependence determines each of the values of the sequence or function, while the second ensures the growth or decrease of the values, allowing the limit to be finite or infinite.

The term 'sufficiently large' might not be explicitly included in all limit definitions, yet it is frequently utilised. Even though it is not inherent in the chosen definition, its consideration is pivotal for generalising the findings of this study to other definitions. 'Sufficiently large' denotes that the specified number surpasses any other number meeting particular conditions.

We must differentiate between two infinite processes in the notion of limit: one for the independent variable and one for the dependent variable. Both of them are associated with potential infinity since there is a never-ending growth process. In addition, actual infinity arises when we consider the sequence and its limit as an infinite numerable set. However, it is not possible to find actual infinity in the limit of functions because the cardinal of the set is non-numerable.

Some types of limits consider the concept of dimensioning. For sequences, two dimensions are considered: the sequence is bounded at the bottom by a term of the sequence, and we can observe the bounding or not of the sequence by the real value starting from a certain term. For functions, only bounding will be taken into account as there is no first term (see Table 4).

TABLE 4: Mathematical comparison between the three types of limits.

Variable	Finite limit of a function at a point	Finite limit of a sequence	Infinite limit of a sequence
Dependency	• Independent variable: $\{x \to f(x)\}$ • Dependent variable: $\{\varepsilon \to \delta(\varepsilon)\}$	• Independent variable: $\{n \to an\}$ • Dependent variable: $\{\varepsilon \to N\}$	• Independent variable: $\{n \to an\}$ • Dependent variable: $\{H \to v\}$
Infinite processes	• Approximation in the independent variable • Approximation to the limit by upper and lower values • Infinite processes continuous	• No approximation in the independent variable • Approximation to the limit by upper or lower or values • Discrete infinite processes	• No approximation in the independent variable • No approximation in the dependent variable • Discrete infinite processes
Types of infinity	• Infinite potential absent • Actual infinity absent	• Present potential infinity • Numerable present infinity	• Present potential infinity • Numerable present infinite
Annotation	• Independent variable bounded • Dependent variable bounded	• Independent variable not bounded • Dependent variable bounded	• Independent variable not bounded • Dependent variable not bounded
Intuition of the continuity of the interval	• It is a requirement	• It is not a requirement	• It is not a requirement

The most significant differences are to be found in the infinite processes and in the intuition of the continuity of the interval. The infinite processes of the dependent variable, in the case of the infinite limit, do not approach any real number in the dependent variable, a fact that implies the differentiation in the non-boundedness of this variable. In contrast, only intuition of the continuity of the interval is needed for the case of functions. Moreover, there is a difference in dependence, where ε tends to zero and H tends to infinity, a fact that is related to infinite processes. Finally, of course, there are differences in the intuition of interval continuity.

Phenomenological results

From an intuitive approach, the phenomenon of Intuitive Double Approximation (i.d.a.) is observed in the case of the finite limit of a function at a point, and the phenomenon of Intuitive Simple Approximation (a.s.i.) characterised for the finite limit of a sequence seems to be split into two phenomena, having differentiated the limit +∞ and -∞, obtaining the phenomena Unbounded Intuitive Growth, u.i.-g, and Unbounded Intuitive Decrease, u.i.-d, for the case of the infinite limit of a sequence.

From a formal approach, there is an analogy between the phenomena of One Way and Return in Functions (o.w.r.f.), One Way and Return in Sequence (o.w.r.s.), and One Way and Returned in Sequences with an Infinite Limit (o.w.r.s.i). In all three cases, there is feedback that is manifested by observing the one-way and round-trip processes together, and interpreting and applying the processes included in each of the definitions, necessitating the construction of a univocal function for the function, or for each sequence and that relates elements of the y axis and of the x axis (see Figure 4).

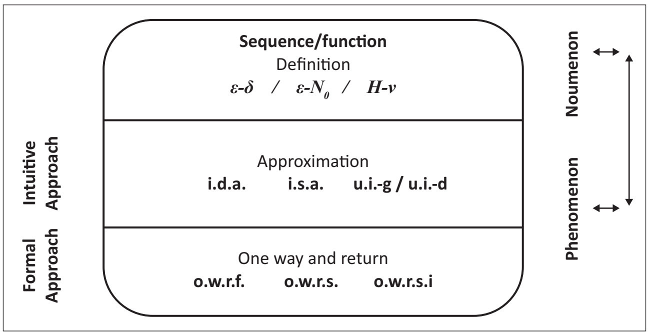

FIGURE 4: Phenomenological comparison.

Textbooks results

Given the varying number of textbooks analysed for each limit due to the appearance of characterised phenomena, we present the average number of phenomena observed in each textbook. Furthermore, the two intuitive phenomena related to the infinite limit of a sequence have been jointly considered (see Table 5 and Figure 5).

TABLE 5: Average number of intuitive phenomena in each textbook.

Representation system-Format	Finite limit of a function at a point	Finite limit of a sequence	Infinite limit of a sequence
v-d	0.64	0.23	0.80
v-e	1.11	0.87	0.91
g-d	0.07	0.00	0.00
g-e	0.50	1.20	0.34
t-d	0.00	0.00	0.00
t-e	0.79	0.40	0.20
s-d	0.04	0.00	0.00
s-e	0.00	0.00	0.00
Subtotals	3.15	2.70	2.25

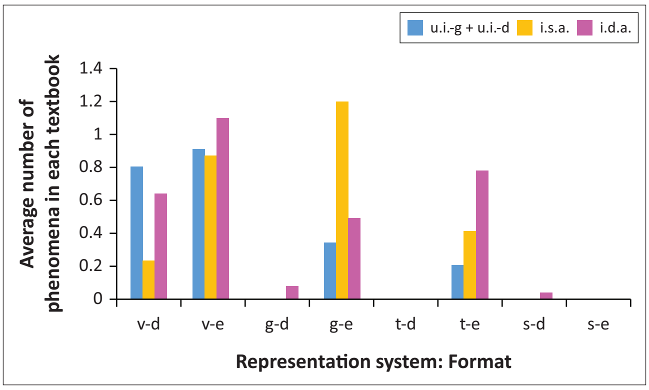

FIGURE 5: Comparison of the intuitive approach in textbooks.

Based on the analysis of intuitive phenomena, we can establish the following relationships:

- The intuitive phenomena of the infinite limit of a sequence primarily prevail within the verbal representation system. Specifically, the representation and example format (v-e) exhibits the highest incidence for the phenomena associated with the infinite limit of a sequence.

- The intuitive phenomena related to the finite limit of a sequence predominantly appear within the graphical representation system, particularly in the example format (g-e), which demonstrates the highest incidence among the representations and formats used.
- The intuitive phenomenon of the finite limit of a function at a point predominates in the verbal representation systems and example format (v-e), and tabular and example format (t-e), the former being the one with the highest occurrence of its own phenomenon. It should be noted that this phenomenon is the only one identified in the graphic and symbolic representation systems and definition format (g-d and s-d).
- None of the phenomena has been identified in the tabular representation system and definition format (t-d), nor in the symbolic representation system and example format (s-e).

In the comparison of formal phenomena, the issue of intuitive phenomena did not arise. There exists a phenomenological correspondence among the three limits, each following what is known as the 'back and forth' process. Once again, this comparison considers all phenomenological codes, incorporating both the system of representation and the average number of phenomena in each book's format (see Table 6 and Figure 6).

Consequently, we can establish the following relationships on a formal approach:

- The formal phenomenon associated with the infinite limit of a sequence predominantly appears within the verbal representation system, encompassing both formats (v-d and v-e), and stands as the representation format with the highest frequency for this phenomenon.
- The formal phenomenon of the finite limit of a sequence predominates in the symbolic representation system and example format (s-e), with the s-e representation being the one that has the greatest occurrence for the phenomenon itself.
- In the graphical representation system, the formal phenomenon of the finite limit of a function at a point exhibits the highest frequency, observed in both examples and definitions. It is important to note that within this representation system, the formal phenomenon of the infinite limit of a sequence does not appear in the example format.
- None of the phenomena has been identified in the tabular representation system and definition format (t-d), and only the formal phenomenon of the finite limit of a function at a point has been identified for the example format, almost tokenistically.

TABLE 6: Average number of formal phenomena in each textbook.

Representation system-Format	Finite limit of a function at a point	Finite limit of a sequence	Infinite limit of a sequence
v-d	0.75	0.83	1.37
v-e	0.43	0.43	1.00
g-d	0.43	0.23	0.09
g-e	0.29	0.23	0.00
t-d	0.00	0.00	0.00
t-e	0.00	0.03	0.00
s-d	0.61	0.57	0.29
s-e	0.54	1.23	0.31
Subtotals	3.05	3.55	3.06

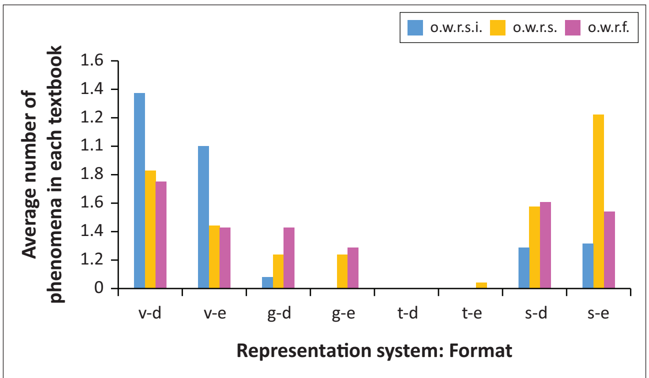

FIGURE 6: Comparison of the formal approach in textbooks.

Conclusion

The initial analysis enabled the establishment of a mathematical comparison among the definitions of the three limits under study. This comparison utilised the definitions outlined in Spivak's manuals (1994) for the finite limit of a function at a point and the finite limit of a sequence, as well as in Linés's work (1983) for the infinite limit of a sequence. These distinctions focused on various mathematical concepts, including dependence, infinite processes, types of infinity, boundedness, and the intuitive understanding of interval continuity.

Subsequently, in the second analysis, a comparison of the phenomena was conducted, as defined by Freudenthal (1983). Previous studies (Arnal-Palacián, 2019; Claros, 2010; Sánchez, 2012) have highlighted the importance of differentiating between the formal and intuitive approaches, as their results differ. When considering the formal approach, there is correspondence between the three phenomena characterised for the three types of limits. In the intuitive approach, there is correspondence between the phenomena characterised for the finite limit of a function at a point and the finite limit of a sequence, but there are certain differences for the infinite limit of a sequence, as it seems to split into two phenomena when dealing with limits approaching positive and negative infinity separately. Despite these differences, there is a common phenomenological structure: on the one hand, we have the intuitive phenomena, and on the other, the formal phenomena for each of the sequences, with the latter guaranteeing the existence of the limit (whether finite or infinite).

Finally, in the analysis of textbooks, we were able to identify four systems of representation: verbal, graphic, symbolic, and tabular (Janvier, 1987), and two formats: example and definition. All of these systems were identified except for the tabular representation system in the definition format. It is noteworthy that the three limits have the highest frequency of occurrence in the verbal representation system and definition format, despite the fact that the use of verbal terms is often considered a difficulty in understanding the notion of limit (Tall & Vinner, 1981). These results differ from the practices of both in-service teachers (Sánchez, 2012) and pre-service teachers (Arnal-Palacián et al., 2022; Arnal-Palacián & Claros-Mellado, 2022) in the classroom.

As a future perspective of this study, we plan to extend our analysis to other types of limits commonly taught in the classroom, horizontal asymptotes and vertical asymptotes. We also aim to develop a teaching sequence that specifically addresses all of these types of limits. This will provide written responses from students highlighting phenomenological features that emerge in their presentation.

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Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

All authors, M.A-P., F.J.C-M. and M.T.S-C., have sufficiently contributed to the study and agreed with the results and conclusions.

Ethical considerations

This article followed all ethical standards for research without direct contact with human or animal subjects.

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Data availability

Data sharing is not applicable to this article as no new data were created or analysed in this study.

Disclaimer

The views and opinions expressed in this article are those of the authors and do not necessarily reflect the official policy of position of any affiliate agency of the authors.

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