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Learners' algebraic and geometric connections when solving Euclidean geometry riders

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In this article, we explored Grade 11 learners' algebraic and geometric connections when solving Euclidean geometry riders. A qualitative interpretive case study design was followed. Thirty Grade 11 learners from a non-fee-paying secondary school in the Capricorn North district of South Africa were conveniently sampled to participate in this study. Data were collected through learners' responses to classwork, homework exercises, and task-based interviews. Data were analysed thematically. The findings revealed that to solve Euclidean geometry riders successfully, learners need to establish the feature connections embedded in the given figure or diagram. The ability to make feature connections provides a point of departure in the solution process of a geometric problem.

Contribution: Once the feature connection is established, other connections will subsequently emerge. In addition, the reversibility connections become a form of feature connection when solving Euclidean geometry riders. Therefore, we recommend that mathematics teachers emphasise and use mathematical connections in their daily teaching of mathematics.

Keywords: algebraic connections; geometric connections; deductive thematic analysis; Euclidean geometry riders.

Introduction

This study explored Grade 11 learners' algebraic and geometric connections when solving Euclidean geometry riders. A Euclidean geometry rider is a geometric problem based on a set of theorems, definitions and axioms (Giannakopoulos, 2017). Solving Euclidean geometry riders requires learners to connect their algebraic and geometric concepts (Kemp & Vidakovic, 2021). South African learners grapple with solving Euclidean geometry problems because of the inability to integrate geometric and algebraic knowledge (Machisi, 2021). In Euclidean geometry, concepts are represented by axioms, definitions, theorems, and proofs (Denbel, 2015, Madzore, 2017). Solving Euclidean geometry problems helps learners to develop logical and deductive reasoning skills, which help them to expand their mental and emotional capacities (Liu et al., 2015). According to Pavlovičová and Bočková (2021), learning geometry improves learners' geometric thinking. Therefore, learners need to be taught Euclidean geometry to develop their conceptual knowledge and analytical skills (Mamali, 2015). Furthermore, solving Euclidean geometry problems requires learners to apply their visualisation skills. Knowledge of arithmetic and algebraic concepts is essential in solving Euclidean geometry problems (Suwito et al., 2016). When solving Euclidean geometry problems, learners interact with shapes in different orientations (Siyepu & Mtonjeni, 2014).

South African learners in Grade 11 are expected to solve Euclidean geometry riders and prove theorems as part of their coursework (Department of Basic Education [DBE], 2011). Euclidean geometry riders are integrated into other mathematics concepts, such as trigonometry, coordinate geometry, and algebra (Denbel, 2015). Furthermore, riders contain different mathematical concepts, for example the congruency of triangles, and properties of parallel lines (Fauzi, 2015). As such, learners need to combine different mathematical content knowledge and procedures when solving Euclidean geometry riders, which also helps them to develop conceptual knowledge (In'am, 2016). According to Govender (2014), learners should connect theorems within Euclidean geometry, and apply concepts from other branches of mathematics such as algebra, trigonometry and analytical geometry. When solving and proving Euclidean geometry riders, learners are required to apply knowledge of theorems and properties of shapes to formulate algebraic equations to solve the problems and interpret them geometrically (Pilgrim & Bloemker, 2016). Therefore, this indicates that learners' competence in proving and solving geometry riders depends on their abilities to connect and integrate algebraic and geometric concepts and processes during the solution process (Sialaros & Christianidis, 2016).

The ability to connect algebraic knowledge and knowledge of Euclidean geometry is an essential prerequisite skill for solving geometric riders as well as developing conceptual integration during problem-solving (Pilgrim & Bloemker, 2016). It is therefore necessary to look at the algebraic and geometric connections that Grade 11 learners make when solving Euclidean geometry riders.

The ability to make connections is an essential skill for solving Euclidean geometry riders successfully (Reddy, 2015). Learners who possess this ability can integrate or identify properties of figures within representations. Mathematical connection refers to the skill of making interrelationships among mathematical concepts, skills, and as well as relating ideas to real-world situations and other related topics (Haji et al., 2017). Learners who possess this skill view mathematics as a complete entity, not a separate subject of distinct concepts (Egodawatte & Stoilescu, 2015). There are two categories of mathematical connections, namely internal and external connections (Baiduri et al., 2020). Internal connections are interrelationships between mathematical topics, mathematical processes and procedures (Baiduri et al., 2020). External connections are interrelationships between mathematics and other subjects in the curriculum as well as everyday life (Ayunani & Indriati, 2020). Thus, mathematical connections provide learners with a broader and more holistic understanding and view of mathematics (Ndiung & Nendi, 2018). Learners learn about the properties of shapes and theorems of Euclidean geometry and then summarise these concepts algebraically using equations (Pilgrim & Bloemker, 2016). When learners possess mathematical connection skills, they will be able to successfully solve Euclidean geometry riders.

Several researchers revealed that learners experienced some challenges when solving Euclidean geometry riders. Makonye and Fakude (2016) reported that learners incorrectly apply Euclidean geometry theorems during problem-solving. The South African National Senior Certificate Diagnostic Report showed that learners had trouble relating prior knowledge to concepts of Euclidean geometry, visualising diagrams and solving problems (DBE, 2020). Furthermore, Ngirishi and Bansilal (2019) reported that South African learners fail to make correct constructions when proving theorems; they incorrectly apply Euclidean geometry theorems and figure properties (DBE, 2020). Learners have misconceptions about geometric concepts because they rely on the physical appearance of the figures, an inability to associate geometric properties with each other, overgeneralisation, and memorisation (Zuya & Kwalat, 2015). In addition, learners struggle to incorporate or connect other mathematical concepts and struggle with writing algorithms correctly when solving riders (DBE, 2020; Luneta, 2015). This indicates that learners experience challenges in connecting geometric and algebraic knowledge when proving and solving riders. Thus, in this article, we explored Grade 11 learners' algebraic and geometric connection skills for solving and proving Euclidean geometry riders.

Literature review

The integration of algebraic and geometric concepts in solving Euclidean geometric problems has been a topic of significant interest in mathematical education. This approach leverages the strengths of both algebra and geometry, fostering deeper understanding and more versatile problem-solving skills. Integrating algebraic and geometric concepts helps students develop a more comprehensive set of tools for solving problems. Algebra provides a systematic approach to solving equations, while geometry offers visual intuition and spatial reasoning. The integration of the two problem-solving skills provides learners with a versatile approach to tackling geometry problems. Studies have shown that learners who are proficient in both areas perform better in complex problem-solving tasks (Fuson et al., 1997).

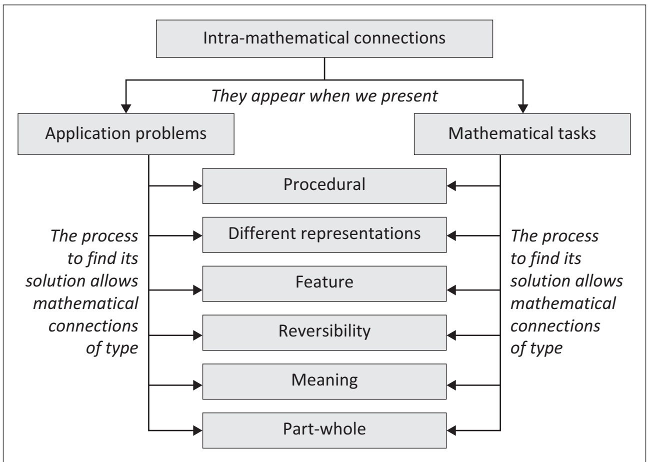
Euclidean geometry is a significant topic for developing mathematical skills. However, learner performance in geometry remains a concern in many countries (Mosia et al., 2023). In South Africa, the teaching and learning of Euclidean geometry has been identified as one of the topics that is a challenge for both teachers and learners (Giannakopoulos, 2017). This observation suggests the need for an urgent intervention to seek an alternative approach to the teaching and learning of Euclidean geometry (Jojo, 2015). Proficiency in using mathematical connections is an important mathematical skill, and learners should embrace it as a tool for solving mathematical problems. Learners with good mathematical connection skills have high success rates in solving mathematical problems. On the other hand, learners with poor mathematical connection skills struggle with solving mathematical problems. Learners' low mathematical connections efficacy impacts on learners' failure in solving mathematical problems. The learners' ability to solve mathematical problems is measured by the number of mathematical connections they are able to make as they are solving the problem. When learners connect mathematical ideas, their understanding is deeper and more lasting, and they come to view mathematics as a coherent whole. By solving mathematical problems, learners gain ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that serve them well outside the mathematics classroom. When learners can connect mathematical ideas, their understanding of mathematics becomes deeper and more durable (Ayunani & Indriati, 2020). Mathematical tasks that involve the relationship between mathematical ideas within a topic and between topics train learners to improve their mathematical connection abilities.

In the South African curriculum for the Further Education and Training (FET) Phase, solving Euclidean geometric problems involves leveraging various mathematical connections and concepts. When solving Euclidean geometry riders, which are essentially challenging geometry problems, integrating algebraic and geometric connections can significantly enhance problem-solving efficiency and depth. By integrating algebraic and geometric approaches, learners can more effectively solve Euclidean geometry riders, deepening their understanding and improving their problem-solving skills. In Euclidean geometry, riders are challenging problems that require a deep understanding of various theorems and their applications. Solving these problems often involves connecting multiple geometric theorems, allowing for a comprehensive approach to complex problem-solving (Giannakopoulos, 2017). By connecting these various mathematical concepts and methods, one can approach Euclidean geometric problems from multiple angles, making the solution process more robust and comprehensive.

Theoretical framework

The mathematical connection framework of GarcÍa-GarcÍa and Dolores-Flores (2018) guided this study. The mathematical connections framework consists of the following tenets: procedural, part-whole, different representations, reversibility, meaning, and feature. Figure 1 shows the theoretical framework that guided this study.

The mathematical connection framework in Figure 1 shows the mathematics connections that can be used during problem-solving. These connections include procedural, different representations, features, reversibility, meaning, and part-whole (GarcÍa-GarcÍa & Dolores-Flores, 2018). This study focused on exploring Grade 11 learners' algebraic and geometric connections when solving Euclidean geometry riders.



Source: Adopted from GarcÍa-GarcÍa, J., & Dolores-Flores, C. (2018). Exploring mathematical connections of pre-university learners through tasks involving rates of change. *International Journal of Mathematical Education in Science and Technology*, *50*(3), 369–389. https://doi.org/10.1080/0020739X.2018.1507050

FIGURE 1: Framework to study mathematical connections.

Procedural connections

Procedural connections are made when learners use formulas, algorithms, and rules to solve a mathematics problem (GarcÍa-GarcÍa & Dolores-Flores, 2018). Procedural connections also include learners' explanations and justifications for using a particular formula or procedure to solve a mathematical problem (GarcÍa-GarcÍa & Dolores-Flores, 2018). Recently, GarcÍa-GarcÍa and Dolores-Flores (2021) defined a procedural connection as a mathematical connection in a situation, in which if learners have or identify concept A, then B automatically becomes a procedure to get the solution. For example, if a learner identifies that a triangle is right-angled, then the procedure for solving it can involve the application of the Theorem of Pythagoras and trigonometric ratios. In proving and solving riders, learners identify the applicable theorems and corollaries from the given geometric diagrams.

Different representations connections

Different representation refers to presenting a mathematical idea in an equivalent or alternative way (GarcÍa-GarcÍa & Dolores-Flores, 2018). An alternative representation is when learners can present a mathematical concept in more than one form of representation (Rodriguez-Nieto et al., 2020). Different forms of representation take the form: algebraic-graphic, verbal–graphic, algebraic-verbal, etc. Alternative different representations connections are observed when learners present theorems as algorithms and when learners present theorems diagrammatically. For example, the theorem stated as 'Angle subtended by an arc at the centre is twice the angle subtended by the same arc on the circumference in the other segment' can be represented algebraically or diagrammatically. Figure 2 presents this theorem.

Equivalent different representations connections, on the other hand, appear within the same register (for example algebraic-algebraic); the focus is more on simplifying the same representation. For example, the algebraic function $f(x) = x^2 + 5x + 6$ is equivalent to $f(x) = (x + 3)(x + 2)$. Equivalent different representations connections are made when learners

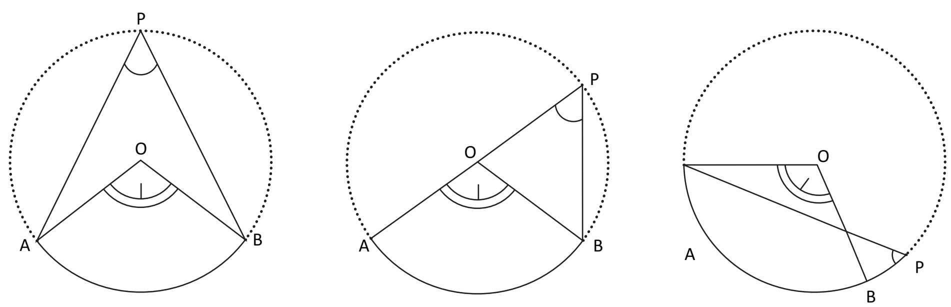


FIGURE 2: The angle at the centre is twice the angle at the circumference.

can present theorems in different forms within one register. For example, the theorem presented in Figure 2 can be written as $\widehat{O}_1 = 2\widehat{P}_1$ or $\widehat{P} = \frac{1}{2}\widehat{O}_1$, and the three pictures in Figure 2 represent the same theorem but in different shapes and orientations.

Feature connections

Feature connections are recognised when learners can identify mathematical concepts in different contexts using their properties (GarcÍa-GarcÍa & Dolores-Flores, 2018). Likewise, feature connections are mostly identified when learners describe or develop descriptions of properties of concepts in terms of the other as a way of differentiating the concepts (Eli et al., 2013). Furthermore, the characteristics of mathematical concepts are used to differentiate them from others (Eli et al., 2013). In the context of Euclidean geometry, the characteristics of mathematical concepts refer to the properties of shapes and their relations (Luneta, 2015). For example, Figure 3 and Figure 4 present different theorems but they have common aspects in terms of shape.

Reversibility connections

According to GarcÍa-GarcÍa and Dolores-Flores (2018), reversibility connections are observed when learners are able to identify the bidirectional relationship between mathematical ideas and concepts. Reversibility connections are evident when learners are able to use concept X to arrive at concept Y and vice versa (GarcÍa-GarcÍa & Dolores-Flores, 2018). Reversibility connections are observed when learners are able to recognise and establish relationships among theorems, corollaries, and converses. Reversibility connections are mostly displayed when learners are supposed to prove Euclidean geometry riders, for example 'the angle between a tangent and a chord is equal to the angle subtended by the same chord in the other segment'. Thus, when learners are not given a circle but are required to prove that a certain segment is a tangent, they are supposed to identify the two angles they can prove are equal and those angles should be equal.

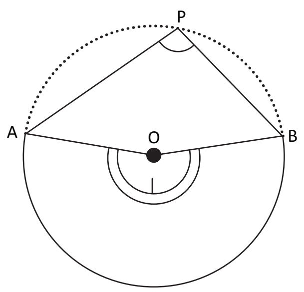


FIGURE 3: The angle at the centre is twice the angle at the circumference.

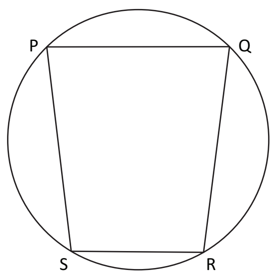


FIGURE 4: Cyclic quadrilateral PQRS.

Meaning connections

Meaning connections are identified when learners can describe mathematical concepts in their own way using relevant reasons and arguments when solving mathematical tasks (GarcÍa-GarcÍa & Dolores-Flores, 2018). Learners give mathematical concepts meaning to differentiate them, to get a sense of what they mean to themselves in different contexts and give them a definition (GarcÍa-GarcÍa & Dolores-Flores, 2021). The difference between feature connections and meaning connections is that mathematical concepts in feature connections are not given definitions while in meaning connections they are given definitions (GarcÍa-GarcÍa & Dolores-Flores, 2018). Proving and solving the riders of Euclidean geometry establish contexts of meaning when learners can identify geometric shapes in different orientations. In addition, meaningful contexts arise when learners can prove the diameter and tangents of the circles that are not drawn. For example, in geometry an exterior angle is one of the angles located outside a geometric figure, but for it to be called an exterior angle it must be on the same straight line as its interior adjacent angle. That is, with the latter, the angle outside a figure will not be called an exterior angle, if it does not fall on the same straight line as the interior adjacent angle.

Part-whole connections

Part-whole connections appear when learners identify the logical relationship between mathematical concepts when solving mathematical tasks (GarcÍa-GarcÍa & Dolores-Flores, 2021). There are two forms of part-whole connections, namely: inclusion and generalisation. Inclusion part-whole connections occur when learners can realise a mathematical concept within another concept (GarcÍa-GarcÍa & Dolores-Flores, 2021). Generalisation part-whole connections occur when learners are able identify general forms of a concept from a specific concept. In Euclidean geometry, generalisation part-whole connections are presented through theorems and their corollaries. Theorems are general statements that learners must conceptualise to the respective geometry diagrams they are presented within a particular question.

Research methods and design

In this article, a qualitative interpretive case study design by Merriam (1998) was utilised. Grade 11 mathematics learners were considered as a case. All 30 Grade 11 learners were conveniently sampled to participate in this study (Cohen et al., 2011). These participants were also purposefully selected as Euclidean geometry is prescribed for them in South Africa. The participants were from a public secondary school in Capricorn North district, South Africa. Participants consisted of 19 girls and 11 boys, and their ages ranged from 16 to 18 years. Data for this study consist of learners' responses to classwork and homework activities. Additionally, data were triangulated using task-based interviews through which learners were asked questions about the connections they were making when solving the given classwork and homework activities. All the task-based interviews were video-recorded and later transcribed. Reflexive thematic analysis was used to analyse the collected data (Braun & Clarke, 2006). The data were coded and grouped into themes according to their commonalities drawn from the mathematical connection framework constructs which comprise procedural, different representation, feature, reversibility, meaning and part-whole connections (GarcÍa-GarcÍa & Dolores-Flores, 2018). These were further classified as either algebraic or geometric connections when finalising the themes that are reported in this study. Data triangulation of classwork and task-based interviews, peer debriefing of all the authors and audit trail of the whole research process were followed to ensure trustworthiness during the analysis process (Nowell et al., 2017). Learners' parents consented to their children's participation in the study. In addition, the participants assented to participate in this study. Pseudonyms were used in place of learners' names. The Limpopo Provincial Department of Education and the selected school gave their permission to conduct the study.

Results

Learners' responses to task-based interviews, classwork and homework activities were analysed thematically in this study. In this study, we selected some of the learners' responses based on the connections they made as they solved the problems. The algebraic and geometric connections that learners made in this study are presented below.

Geometric feature connections leading to algebraic procedural connections

Some learners managed to identify the properties of geometric figures embedded in the rider. These learners made geometric feature connections which led to procedural connections which are algebraic. For example, Learner 26's response to question 3 of class activity 2 is shown in Figure 5.

From Figure 5, the learner started by making feature connections that helped him to determine the length of AB and BC. The learner further applied algebraic connections to determine the correct length of AB. The statements AB = $4x$ and BC = $4x$ show that the learner was able to make feature connections by applying the theorem which states that 'A line drawn from the centre of the circle perpendicular to the chord bisects the chord'. Thus, the learner was able to connect the correct theorem with the diagram through the features of the given diagram. Therefore, the learner was able to make the geometric feature connection by applying the correct theorem to the given diagram. In addition, the learner made a geometric feature connection by applying the Theorem of Pythagoras to determine the radius OC of the circle. The learner managed to substitute correctly, thus making the correct algebraic procedural connection. However, the learner could not get to the solution because of simplification after substitution into the Theorem of Pythagoras. This learner wrote $\sqrt{OC} = 2x$, which means that the learner was taking the square root of each term of the equation. The learner failed to simplify the surds, hence committing an algebraic procedural error. When interviewed about why he wrote that way, the learner responded by saying: '$\sqrt{OC}$ is the square root of OC^2 and $2x^2 + (4x)^2 = 2x$ This learner failed to make connections between the square and the square root concepts which affected the learner's solution process. Therefore, the learner was unable to finish the procedure due to a lack of knowledge of simplifying the surds.

Geometric feature connections leading to algebraic procedural connections

From Figure 6, item 1.1, Learner 7 was able to make geometric feature connections. The learner identified that $\hat{A}_2 = \hat{B}_2$ and managed to justify that statement with the correct reason. The learner was able to realise that OA and OB are radii of the given circle and are equal, hence making triangle ABO an

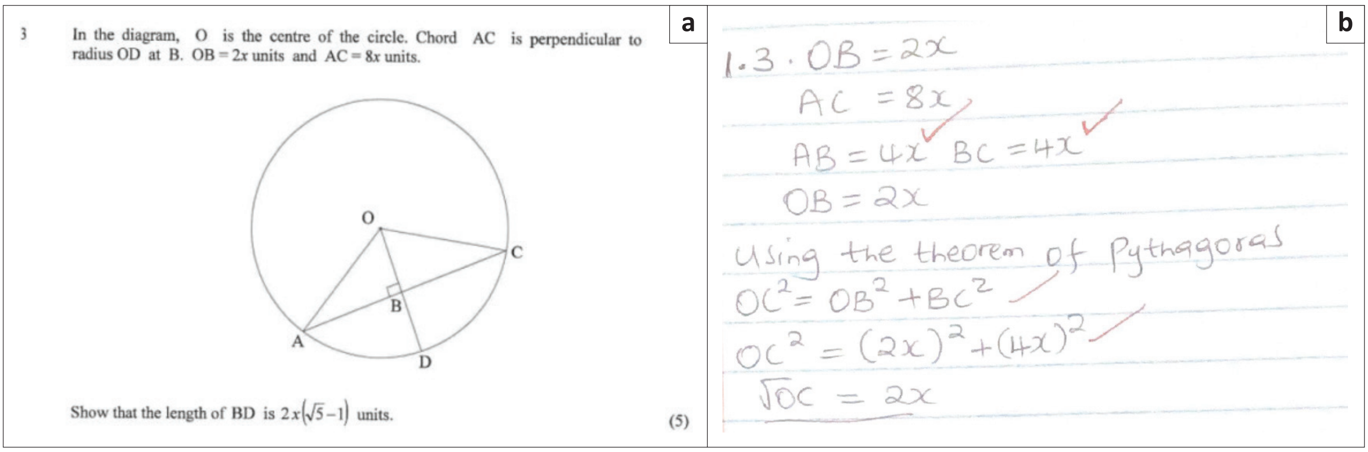


FIGURE 5: Learner 26 response to question 3 of class activity 2: (a) Question and (b) Learner's response.

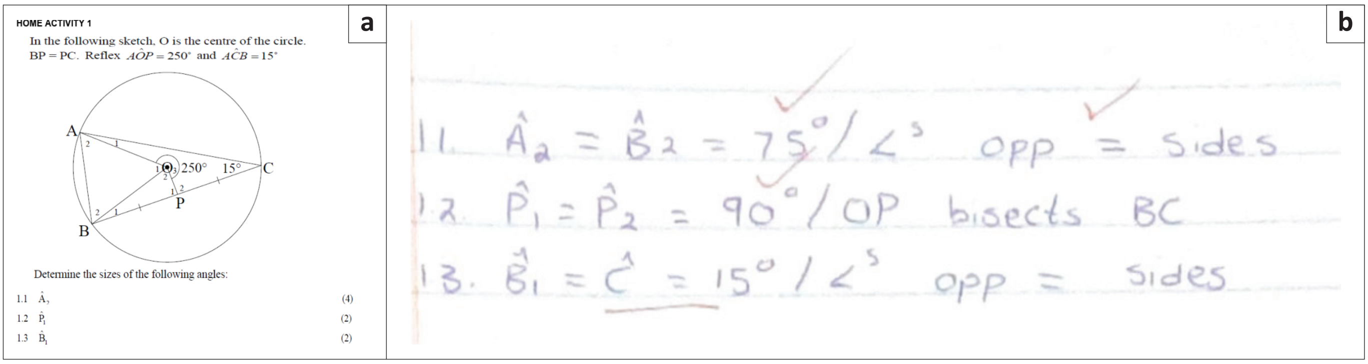


FIGURE 6: Learner 7 response to home activity 1: (a) Question and (b) Learner's response.

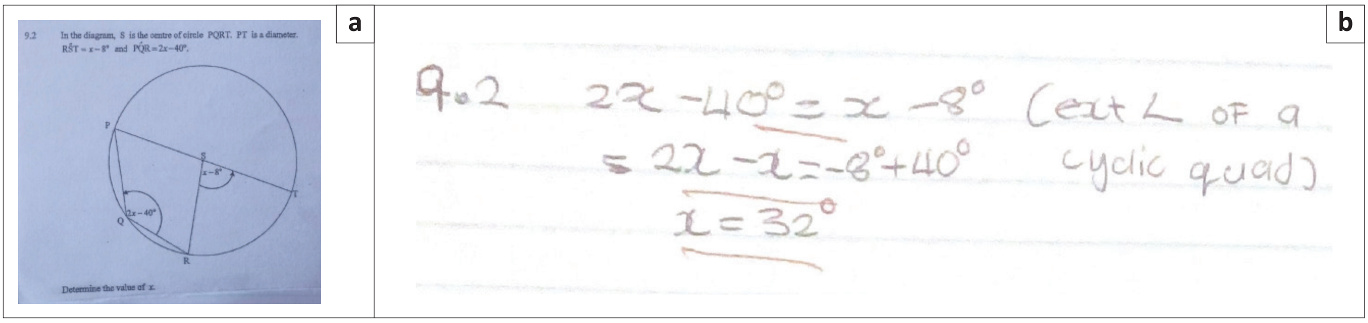


FIGURE 7: Learner 9 response to question 2 of home activity 2: (a) Question and (b) Learner's response.

isosceles triangle. Although the learner did not write all the algebraic steps of determining the size of $\widehat{A}_2$, the feature connections established helped the learner find that the size of $\widehat{A}_2 = 75°$. This also emanated from the learner being able to connect $\widehat{O}_1$ to $A\widehat{C}B$. During the task-based interviews, when asked how she arrived at the answer, the learner responded by saying: 'I applied the theorem of the angle at centre is equal to twice the angle at the circumference. Then this means that $A\widehat{O}B = 30°$. Then the angles of a triangle add up to 180°. Therefore, $180° - 30°$ all divided by $2 = 75°$'. This also indicates that the learner was also able to make algebraic procedural connections and hence managed to find that $\widehat{A}_2 = \widehat{B}_2 = 75°$.

Geometric reversibility becoming feature connections

Generally, learners' responses show that they were able to make reversibility connections that informed correct geometric feature connections. For example, in item 1.2, in Figure 6, the learner noticed that segment BP is equal to PC and managed to identify the size of angle $\widehat{P}_1$ as equal to 90°. When asked why she wrote the reason that way, the learner responded by stating the theorem which states that 'A line drawn from the centre of the circle perpendicular to the chord, bisects the chord'. The learner made the correct feature connection and reversibility connection and got the correct size of angle $\widehat{P}_1$; however, the reasoning during interviews was not correct. She should have stated the reason as 'If a line is drawn from the centre of a circle to the mid-point of a chord, then the line is perpendicular to the chord' which is the corollary of the theorem that she stated as a reason during interviews.

Failure to make geometric feature connections

Most learners failed to make correct geometric feature connections. For example, Learner 7's attempt for item 1.3, in Figure 6, indicates the inability to make geometric feature connections. The learner wrongly stated that angle $\widehat{B}_1$ is equal to angle $\widehat{B}$. In this case, the learner made a wrong geometric feature connection by applying properties of an isosceles triangle in a triangle that is not isosceles. When interviewed the learner stated that 'the two angles are angles opposite equal sides of an isosceles triangle'. When asked which triangle is isosceles, that's when the learner realised that she made a mistake.

In addition, the inability to make correct feature connections was also evident in Figure 7 in which Learner 9' s work on one of the tasks is presented.

Figure 7 reveals that Learner 9 started by equating angles $P\widehat{Q}R$ and $R\widehat{S}T$. The justification for equating the two mentioned angles was given as 'exterior angle of a cyclic quadrilateral is equal to its interior opposite angle'. The learner failed to make a geometric feature connection. According to this learner, any four-sided figure inside a circle is regarded as a cyclic quadrilateral. However, PQRS is not a cyclic quadrilateral. The learner failed to understand that for a quadrilateral to be cyclic all its vertices should be on the circumference of the circle. Although the learner showed strong algebraic connections by being able to group like terms ($2x - x = 80° + 40°$) together, the learner started on the wrong footing by equating angles that are not equal ($2x - 40° \neq x - 8°$). The learner failed to realise that the reflex $P\widehat{S}R$ ($180° + x - 8°$) is the angle at the centre of $P\widehat{Q}R$($2x - 40°$). This learner failed to apply the theorem that states that the

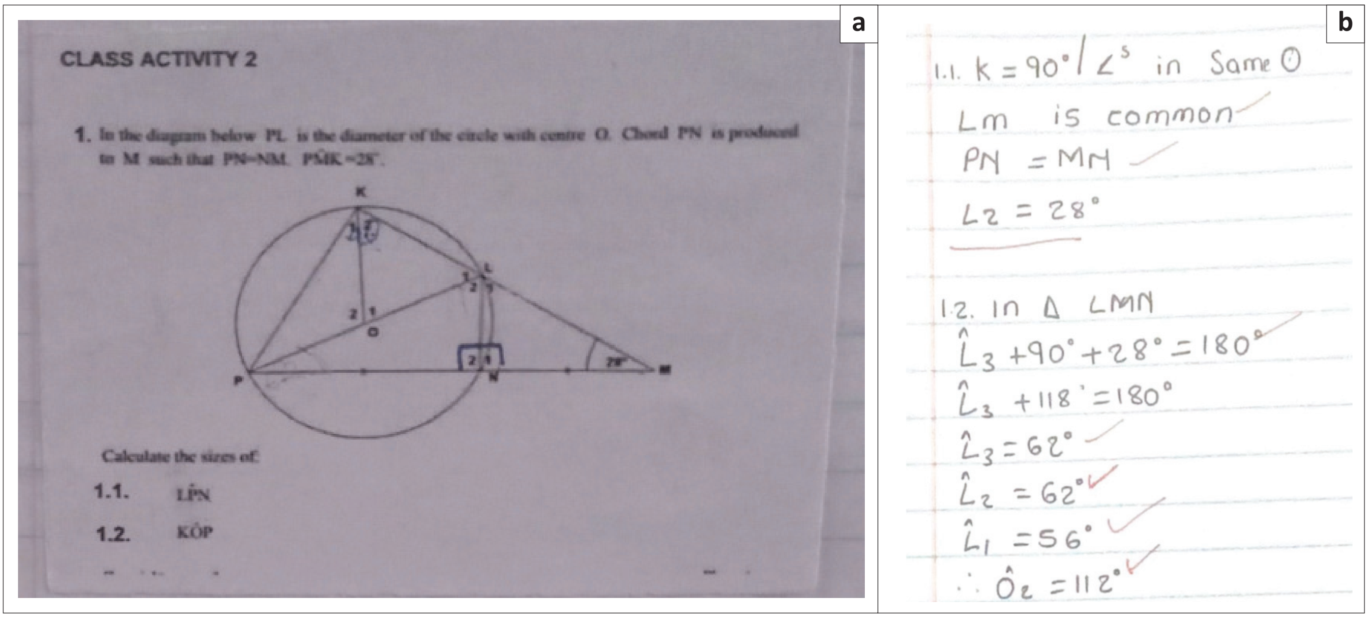


FIGURE 8: Learner 17 responses to class activity 2: (a) Question and (b) Learner's response.

angle at the centre is twice the angle at the circumference of the circle. Hence, this affected the learner's solution process.

Geometric feature connections leading to geometric and algebraic procedural connections

Some learners were able to make correct geometric feature connections which led to geometric and procedural connections when solving Euclidean geometry riders. Figure 8 presents a learner's work on a given question. This question in Figure 8 requires learners to make a transition from geometric feature connections to either geometric or algebraic connections. The two solutions are discussed separately below.

Geometric feature connections leading to geometric procedural connections: From Figure 8, it can be noticed that in an attempt to get the solution of item 1.1, Learner 17 started by stating that $\widehat{K} = 90^{o}$, which was true, and further gave the correct reason for that statement although he made a typo by writing same circle instead of a semi-circle. This indicates that the learner was able to identify the angle subtended by the diameter in the given diagram and made the geometric feature connection that it is equal to 90°. The learner realised that to get the correct solution for item 1.1, there is a need to prove congruency in the two triangles: ΔLPN and ΔLMN. Feature connections were identified again when the learner stated that LM is common when proving congruency in the two triangles. The learner did not manage to finish the congruency procedure; instead he concluded by stating that $\widehat{L}_2 = 28°$. This indicates that the learner was equating $\widehat{L}_2$ to $\widehat{M}$. Failure to make geometric procedural connections by writing the last statement of congruency meant the learner did not arrive at the final solution, that is the size of angle $L\widehat{P}N$. The learner was confused by starting with the statement that $\widehat{K} = 90°$ because $\widehat{K}$ is not an interior angle of any of the triangles, ΔLPN or ΔLMN. In trying to gain insight into the learner's responses, the researcher asked him the following question: 'Why do you start by writing that $\widehat{K} = 90°$?' The learner responded by saying 'I was trying to prove congruency in the two triangles. So, since we are given that PL is a diameter it means that $\widehat{K} = 90°$'. The learner made a correct feature connection; however, this was not necessary to solve the given problem. Instead, he should have used angles $\widehat{N}_1$ and $\widehat{N}_2$ because the two angles are equal. Thus, Learner 17's procedural and feature connections were partially correct because everything given on 1.1 was correct, but there were some important statements left out which could have helped the learner to reach the correct solution.

Geometric feature connections leading to algebraic procedural connections: On item 1.2 in Figure 8, Learner 17 made correct geometric features and algebraic procedural connections in an attempt to solve the problem. The learner made a geometric feature connection by using the sum of interior angles of a triangle to determine the size of angle $\widehat{L}_3$. He then used angle $\widehat{L}_3$ to determine the size of angle $\widehat{L}_2$ because the two angles are of equal since triangles ΔLPN and ΔLMN are congruent. Furthermore, the learner used angles $\widehat{L}_2$ and $\widehat{L}_3$ to determine the size of angle $\widehat{L}_1$ because the three angles are on a straight line adding up to 180°. Therefore, the learner made the correct geometric feature connections as well as algebraic procedural connections. The learner made another geometric feature connection when determining the size of angle $\widehat{L}_1$, because $\widehat{L}_1$ is half of angle $K\widehat{O}P$ according to the theorem 'angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc on the circumference'. The learner applied this theorem to get the size of angle $K\widehat{O}P$, which was the one required.

Discussion

This study explored Grade 11 learners' algebraic and geometric connections when solving Euclidean geometry riders. Data were collected from a conveniently selected sample of 30 Grade 11 learners who responded to classwork and homework activities and were interviewed on their responses to the given tasks. Data were analysed using reflexive thematic analysis. From the thematic analysis, this study has established the following.

In this study, we found that feature connections are at the centre of solving geometric riders. For Grade 11 learners to be able to solve Euclidean geometry riders successfully, they need to establish the feature connections embedded in the given figure or diagram. The ability to make feature connections provides a point of departure in the solution process of a geometric problem. Once the feature connection is established, other connections will subsequently emerge. This study established that when solving riders, the feature connections needed to be made are always geometric as they are based on the shapes, lines and theorems in the given rider (Fauzi, 2015). Therefore, failure to make feature connections causes learners to fail to make a breakthrough to rider solutions. This is in line with Ngirishi and Bansilal (2019) who found that when learners fail to make connections between shapes and properties of shapes, they subsequently fail to solve geometric problems. The ability to make the correct feature connections helps learners to proceed with the solution process as identified in some learners in this study. This implies that teachers need to emphasise the skill of making feature connections when teaching solutions to Euclidean geometry riders.

It was further established in this study that learners who were able to make correct geometric feature connections managed to proceed with the solution process as identified in some learners' work. It is established that making correct feature connections leads to making procedural connections which could either be algebraic or geometric. For algebraic procedural connections, this study discovered that after identifying the properties of the given figure, learners were able to apply algebraic processes, for example simplification, needed to solve the problem. This finding is consistent with Suwito et al. (2016) who pointed out that solving Euclidean geometry problems requires an understanding of algebraic and arithmetic concepts. However, some of the learners did not manage to find the correct solution to the problem due to a lack of algebraic procedural connections. This indicates that these learners were unable to summarise the geometric concepts algebraically using equations (Pilgrim & Bloemker, 2016). For geometric procedural connections, it has been discovered that learners were able to make geometric procedural connections when applying congruency after identifying the properties of the given triangle. This result concurs with Fauzi (2015) who indicated that for learners to solve Euclidean geometry riders successfully they need to connect different geometric concepts such as the congruency of triangles. These results also indicate that learners were unable to complete the geometric procedure as they didn't manage to finish the congruency of the triangle. This might be due to a lack of identification of the geometric properties of the given triangles.

In addition, this study found that when solving Euclidean geometry riders, geometric reversibility connections become a form of feature connection. This has been identified when some learners gave reasons about a theorem instead of stating the corollary of that theorem. This indicates that switching between feature and reversibility connections in Euclidean geometry does not hinder learners' solution process when solving riders.

Conclusion

The findings of this study indicated that making geometric feature connections is the starting point of solving Euclidean geometry riders. Therefore, we recommend future studies to be conducted focusing on geometric feature connections as a base for solving Euclidean geometry riders. Furthermore, the findings show that reversibility connections become a form of feature connections and do not hinder learners' solution process. Therefore, we conclude that learners need to be taught to identify all the geometric features embedded in a given rider first before attempting to solve it. Failure to make correct feature connections results in learners' inability to solve the given Euclidean geometry riders. Therefore, we recommend that learners be equipped with sufficient and appropriate mathematical connections during the teaching and learning of Euclidean geometry. In addition, learners should be taught the skill to identify the feature connections embedded in the given rider and this will help them during the problem-solving process of Euclidean geometry. The feature connections will enable learners to make correct algebraic and geometric connections.

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Competing interests

The authors declare that they have no financial or personal relationships that may have inappropriately influenced them in writing this article.

Authors' contributions

H.B., K.M. and P.M. have sufficiently contributed to the study. This work stems from the first author's Master's dissertation. The second and the third authors were supervisors of the dissertation.

Ethical considerations

This research adhered to the relevant ethical considerations according to the Turfloop Research Ethics Committee of the University of Limpopo. Ethical clearance number: TREC/356/2022: PG.

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Data availability

Data presented in the findings are available upon request from the corresponding author.

Disclaimer

The views and opinions expressed in this article are those of the authors and are the product of professional research. It does not necessarily reflect the official policy or position of any affiliated institution, funder, agency, or that of the publisher. The authors are responsible for this article's results, findings, and content.

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