On Taylor-Proudman columns and geostrophic flow in rotating porous media

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Abstract

Taylor-Proudman columns are a well-known phenomenon in rotating flows in pure fluids (non-porous domains). A theoretical formulation of the problem of incompressible fluid flow in rotating porous media is presented. The criteria for the relative significance of different terms in the equations are identified leading to a formulation which is based on the traditional Darcy's law but extended to include the Coriolis and centrifugal terms resulting from rotation. Finally a proof is provided showing that Taylor-Proudman columns exist in porous media as well. This occurs in the limit of small values of the porous media Ekman number. The corresponding consequences are that a stream function exists in this otherwise three-dimensional flow and this stream function and the pressure are the same in the limit of high rotation rates. This type of geostrophic flow means that isobars represent stream-lines at the leading order for small values of Ekman number.

Nomenclature

Latin symbols

$\hat{\mathbf{e}}_z$	a unit vector in the vertical direction
$\hat{\mathbf{e}}_\omega$	a unit vector in the direction of the imposed angular velocity
$\hat{\mathbf{e}}_g$	a unit vector in the gravity direction
$\hat{\mathbf{e}}_n$	a unit vector normal to the boundary
Ek	the porous media Ekman number defined by eq.(3)
Fr_g	gravity related Froude number, equals $\frac{q_c^2}{g_* l_c}$
Fr_ω	centrifugally related Froude number equals $\left(\frac{q_c}{\omega_c l_c}\right)^2$
h	the bottom topography of the container
$k(x,y,z)$	the dimensionless permeability function
k_0	a reference value of permeability
l_c	a macroscopic characteristic length
p_i	dimensionless pressure
p_r	the dimensionless reduced pressure generalized to include the centrifugal and gravity terms, equals $p_i - \left(\frac{Re_\Delta}{2Fr_\omega}\right)(\hat{\mathbf{e}}_\omega \times \mathbf{X})\cdot(\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \left(\frac{Re_\Delta}{Fr_g}\right)(\hat{\mathbf{e}}_g \cdot \mathbf{X})$
p	rescaled pressure, equals $Ek\, p_r$
r	a coordinate in the radial direction (in a cylindrical system of coordinates)
Re	macroscopic Reynolds number, equals $\frac{q_c l_c}{\nu_0}$
Re_Δ	pore size Reynolds number
q_c	a characteristic filtration velocity
$\mathbf{q}$	the dimensionless filtration velocity, relative to the rotating solid matrix
u	horizontal component of filtration velocity in the x direction
v	horizontal component of filtration velocity in the y direction
w	vertical component of filtration velocity
$\mathbf{X}$	position vector, equals $x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$
x	a horizontal coordinate (in a Cartesian system of coordinates)
y	a horizontal coordinate (in a Cartesian system of coordinates)
z	the coordinate in the vertical direction (in both Cartesian or cylindrical systems of coordinates)

Greek symbols

ϕ	porosity of the porous domain
ω_c	the angular velocity of rotation
ν_0	the kinematic viscosity
θ	a coordinate in the angular direction (in a cylindrical system of coordinates)
ψ	a stream function defined by eq.(11)

Subscripts

0	reference values
c	characteristic values
$*$	dimensional values

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Introduction

Rotating flows and heat transfer in porous media have a wide spectrum of applications in engineering and geophysics. The food process industry, chemical process industry and centrifugal filtration processes are some of the traditional applications. More explicitly packed bed mechanically agitated vessels are used in the food processing and chemical engineering industries in batch processes. As the solid matrix rotates due to the mechanical agitation, a rotating frame of reference becomes necessary. The filtration velocity is thus measured relative to this rotating frame of reference which is connected to the solid matrix. Other, modern applications emerged recently as a result of using the porous media approach to non-traditional disciplines including some domains in which the solid matrix is subjected to rotation. Among these applications, the flow of liquid in human tissues like the brain or heart, the development of porous turbine blades and cooling of electronic equipment subject to rotation (e.g. a rotating radar) may serve as examples. Vadasz [1] presented a more detailed discussion of these applications. Nevertheless, no reported research could be found on *isothermal* flow in *rotating* porous media. Probably, the main reason behind the lack of interest for this type of flow is that the *isothermal* flow in *homogeneous* porous media following Darcy's law is irrotational. However, for a heterogeneous medium with spatial dependent permeability the flow is not irrotational anymore. An example of flow in a rotating heterogeneous porous medium at high values of Ekman number was presented by Vadasz.[1]

In this paper further results are presented showing theoretically that Taylor-Proudman columns, which are a common phenomenon for rotating flows in pure fluids (non-porous domains), exist in porous media as well in the limit of small values of Ekman number.

Problem formulation

Transport phenomena in porous media are represented by a mathematical model at a macroscopic level. This representation is achieved by averaging over a Representative Elementary Volume (REV) the Navier-Stokes and other transport equations which are valid at the microscopic, pore-size scale. Different approaches for averaging have been proposed by Bear & Bachmat,[2] Whitaker,[3] Barrere, Gipoloux & Whitaker [4] and Du Plessis & Masliyah.[5] Eventually a set of equations is obtained at the macroscopic level which are an extension of Darcy's law to include inertial and other effects. As a consequence of the averaging process new variables are defined, e.g. the filtration velocity $\mathbf{q}$ is the average (over the REV) of the real velocity, and the pressure in porous media is the average (over the REV) of the real pressure. New properties are introduced as well through the averaging process, like the porosity ϕ which is the ratio of the pore volume over the total volume of the porous domain, and the permeability k_0, which has the units of square length, representing in principle at the macroscopic level the effective cross-sectional area of the microscopic flow. A major significance of the averaging approach is that it allows one to obtain theoretically the criteria for neglecting terms in the equations. For example, the porous media Reynolds number Re_Δ defined as $Re_\Delta = ReDa = \frac{q_c k_0}{v_0 l_c}$, controls the validity of Darcy's law. When Re_Δ is kept small the inertial effects are insignificant. However, the relationship between the familiar Reynolds number in pure fluids Re (non-porous domains) and the porous media Reynolds number is given by the multiplying factor Da, which represents a Darcy number and is defined as $Da = \frac{k_0}{l_c^2}$. This is the square of the ratio between a pore-size length scale $\sqrt{k_0}$ and the macroscopic length scale of the problem, l_c. As such, Da is typically very small $(10^{-10} - 10^{-5})$, hence extending the validity of Darcy's regime to include a wide range of Re number values. However, the averaging techniques were traditionally applied to the Navier-Stokes equations in a non-rotating frame of reference. When the porous medium rotates a rotating frame of reference becomes necessary in order to keep the averaged equations valid, since the filtration velocity is defined relative to the solid matrix and the later rotates as a solid body. As soon as a rotating frame of reference is introduced two additional inertial effects should be incorporated in the model, i.e. the centrifugal and the Coriolis accelerations. The criteria for their relative significance is not controlled by Re_Δ alone but by other dimensionless groups, like the Ekman number. If Ekman number is very high $(Ek \to \infty)$ then the Coriolis effect becomes insignificant.

As a result, it was concluded (Vadasz [6]) that the following dimensionless equations govern the incompressible flow in rotating heterogeneous porous media under isothermal conditions.

(i) Continuity equation

$$\nabla \cdot \mathbf{q} = 0 \tag{1}$$

(ii) Darcy's law extended to include the Coriolis term

$$\mathbf{q} = -k\left[\nabla p_r + Ek^{-1}\hat{\mathbf{e}}_\omega \times \mathbf{q}\right] \tag{2}$$

where $\mathbf{q}$ is the dimensionless filtration velocity, p_r is the dimensionless reduced pressure generalized to include the centrifugal and gravity terms, $k(x,y,z)$ is the dimensionless permeability function, $\hat{\mathbf{e}}_\omega$ is a unit vector in the direction of the imposed angular velocity and Ek is the porous media Ekman number defined in the form

$$Ek = \frac{\phi v_0}{2\omega_c k_0} \tag{3}$$

where ϕ is porosity, ω_c is the angular velocity of rotation, k_0 is a reference value of permeability and v_0 is the kinematic viscosity.

Equations (1) and (2) are presented in a dimensionless form where the values of $\frac{v_0}{l_c}$ and $\frac{\mu_0 v_0}{k_0}$ are used to scale the filtration velocity and pressure, respectively, and k_0 is used to scale the permeability function k_*.

The Taylor-Proudman theorem in porous media

Equation (2) can be presented in the following form

$$\left[1 + \frac{k}{Ek}\hat{\mathbf{e}}_\omega \times\right]\mathbf{q} = -k\nabla p_r \tag{4}$$

Multiplying eq.(4) by $[\frac{Ek}{k}]$ and rescaling the pressure in the form $p = Ek\ p_r$ yields

$$\left[\frac{Ek}{k} + \hat{e}_\omega \times\right]\mathbf{q} = -\nabla p \tag{5}$$

Given typical values of viscosity, porosity and permeability one can evaluate the range of variation of Ekman number in some engineering applications. There, the angular velocity may vary from 10 rpm to 10 000 rpm leading to

Ekman numbers in the range from $Ek = 1$ to $Ek = 10^{-3}$. The later value is very small, pertaining to the conditions considered in this paper. Therefore, in the limit of $Ek \to 0$, say $Ek = 0$, and assuming $\hat{\mathbf{e}}_\omega = \hat{\mathbf{e}}_z$ equation (5) takes the simplified form

$$\hat{\mathbf{e}}_z \times \mathbf{q} = -\nabla p \tag{6}$$

and the effect of permeability variations disappears. Taking the 'curl' of equation (6) leads to

$$\nabla \times (\hat{\mathbf{e}}_z \times \mathbf{q}) = 0 \tag{7}$$

Evaluating the 'curl' operator on the cross product of the left-hand side of equation (7) gives

$$(\hat{\mathbf{e}}_z \cdot \nabla)\, \mathbf{q} = 0 \tag{8}$$

Equation (8) is identical to the Taylor-Proudman form for pure fluids (non-porous domains); it thus represents the proof of the Taylor-Proudman theorem in porous media and can be presented in the following simplified form

$$\frac{\partial \mathbf{q}}{\partial z} = 0 \tag{9}$$

The conclusion expressed by equation (9) is that $\mathbf{q} = \mathbf{q}(x, y)$, i.e. it cannot be a function of z. This means that all filtration velocity components can vary only in the plane perpendicular to the angular velocity vector.

Results and discussion

An example of a Taylor-Proudman column

The consequence of the result presented in the previous section can be demonstrated by considering a particular example. Figure 1 shows a closed cylindrical container filled with a fluid saturated porous medium. The topography of the bottom surface of the container is slightly changed by fixing securely a small solid object (see Greenspan [7] for the corresponding example in pure fluids). The container rotates with a fixed angular velocity ω_c. Any forced horizontal flow in the container is expected to adjust to its bottom topography. However since equation (9) applies for each component of $\mathbf{q}$ it applies in particular to w, i.e. $\frac{\partial w}{\partial z} = 0$. But the impermeability conditions at the top and bottom solid boundaries require $\mathbf{q} \cdot \hat{\mathbf{e}}_n = 0$ at $z = h(r, \theta)$ and at $z = 1$, where $h(r, \theta)$ represents the bottom topography. The combination of this boundary condition with the requirement that $\frac{\partial w}{\partial z} = 0$ yields $w = 0$ anywhere in the container. Hence, a flow over the object as described qualitatively in Figure 2 becomes impossible as it introduces a vertical component of filtration velocity. Therefore the resulting flow may adjust around the object as presented qualitatively in Figure 3. However since this flow pattern is also independent of z, it extends over the whole height of the container resulting in a **fluid column** above the object which rotates as a **solid body**. This is a demonstration of a Taylor-Proudman column in porous media, as presented qualitatively in Figure 4.

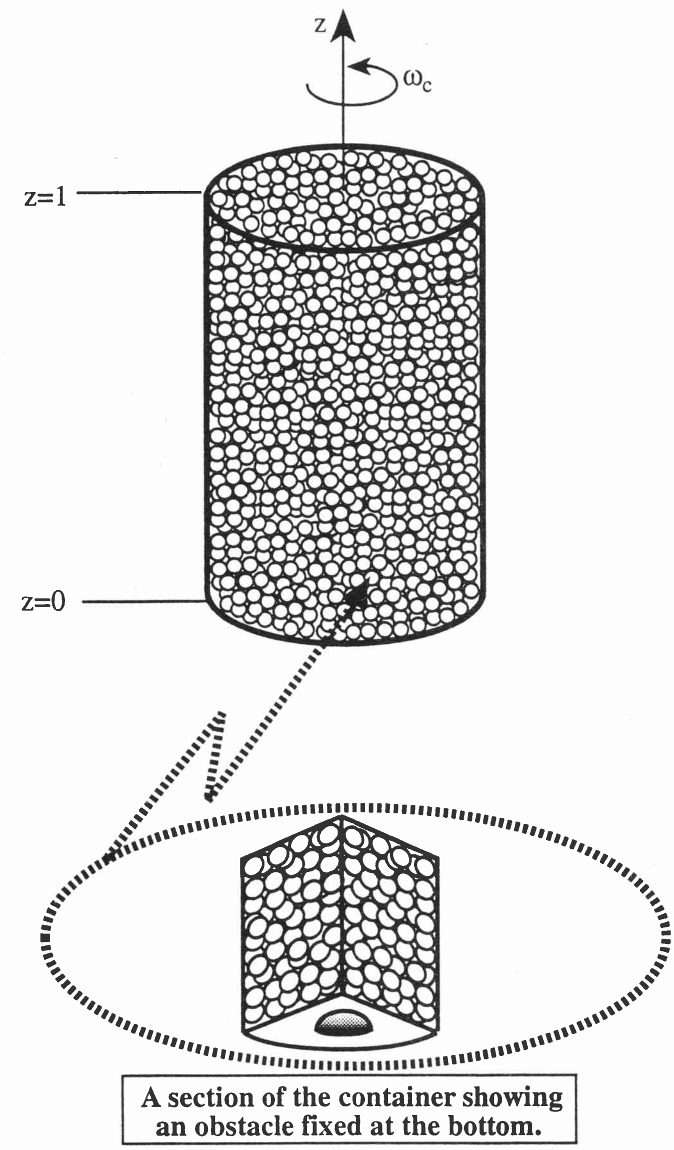

Figure 1 A closed cylindrical container filled with a fluid saturated porous medium.
A solid object is fixed at the bottom.

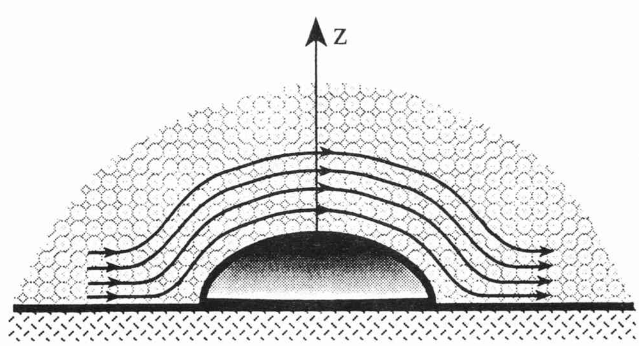

Figure 2 An impossible type of flow over the object.

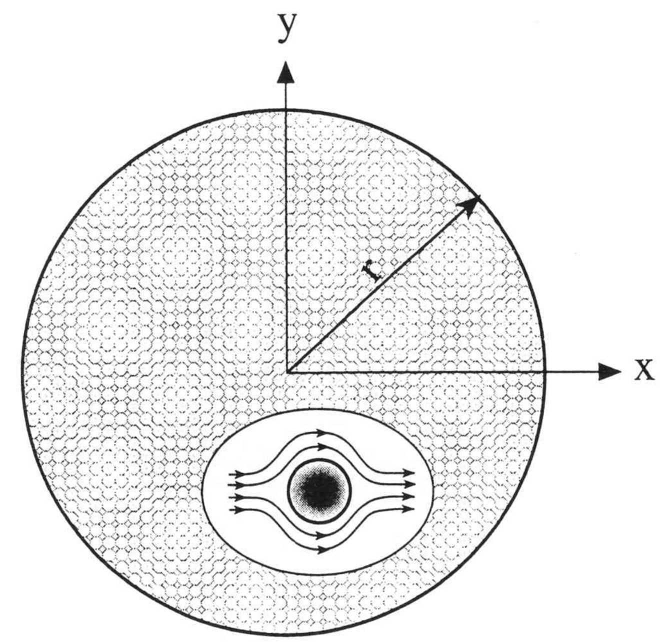

Figure 3 The flow adjusts around the object (as seen from above) and extends at all heights creating a column above the object which behaves like a solid body.

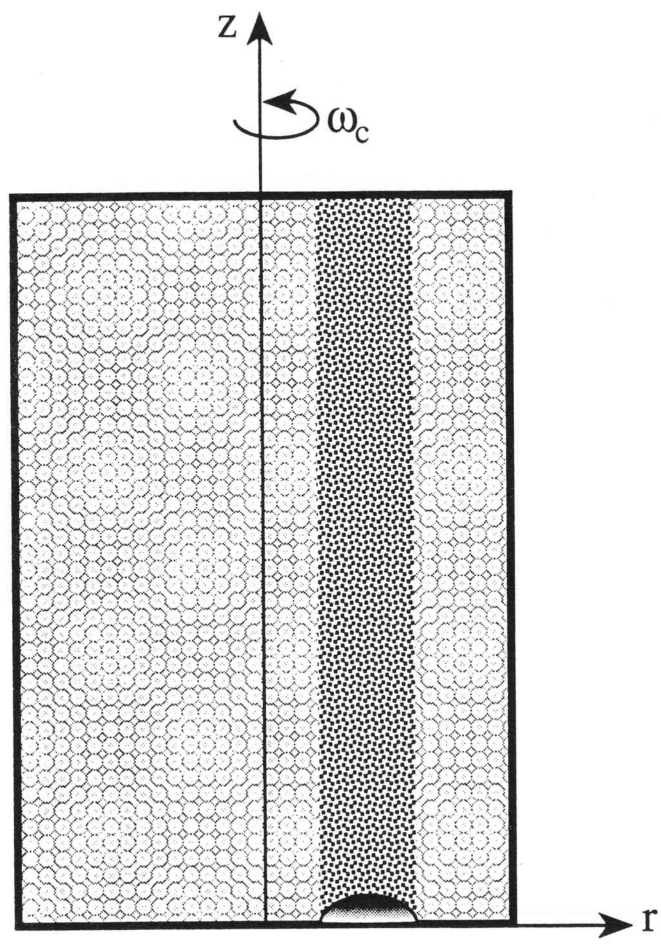

Figure 4 Qualitative description of a Taylor-Proudman column in porous media.

Geostrophic flow in rotating porous media

A further significant consequence of equation (9) is represented by a geostrophic type of flow. It is observed that despite imposing a permeability which is a function of z as well, the flow at high rotation rates, i.e. $Ek \rightarrow 0$, is independent of z. In particular $\frac{\partial w}{\partial z} = 0$, and the continuity equation (1) presented in Cartesian coordinates becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (10)$$

A stream function, ψ, can therefore be introduced for the flow in the $x - y$ plane

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \qquad (11)$$

Through the definition of the stream function ψ, given by equation (11), the continuity equation (10) is identically satisfied. Substituting u and v with their stream function representation given by equation (11) into eq.(6) yields

$$\frac{\partial \psi}{\partial x} = \frac{\partial p}{\partial x} \qquad (12)$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial p}{\partial y} \qquad (13)$$

The conclusion resulting from equations (12) and (13) is that the **stream function** and the **pressure** are the same in the limit of high rotation rates ($Ek \rightarrow 0$). This type of geostrophic flow means that **isobars** represent **streamlines** at the leading order for $Ek \rightarrow 0$.

Conclusions

A theoretical formulation and proof of existence of Taylor-Proudman columns in porous media in the limit of small values of the porous media Ekman number was presented. The corresponding consequence leading to a geostrophic type of flow in porous media was discussed. Experimental confirmation of the theoretical results is recommended despite the practical difficulty of reproduction of experiments from pure fluids to porous domains.

Acknowledgement

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