RADIAL FLOW BOUNDARY LAYER DEVELOPMENT ANALYSIS

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A boundary layer analysis was performed to determine the pressure differential due to frictional effects, and the heat transfer coefficient during turbulent radial fluid flow between two approximately parallel discs or surfaces. The results of the analysis were applied to flow at the inlet of the collector of a solar chimney power plant and a numerical example is presented to show the effect that various independent variables have on the radial pressure and heat transfer coefficient.

Nomenclature

b exponent
c_p specific heat, J/kg-K
d_e hydraulic diameter, m
f friction
g gravitational acceleration, m/s^2
H height, m
h heat transfer coefficient, W/m^2-K
K loss coefficient
k thermal conductivity, W/m-K
m mass flow rate, kg/s
Nu Nusselt number, $h(r_0 - r)/k$
p pressure, N/m^2
Pr Prandtl number, $\mu c_p/k$
r radius, m
Ra Rayleigh number, $g\beta\Delta TH^3/\alpha\text{-}v$
Re Reynolds number , $\rho v d_e/\mu$
T temperature, K
v velocity, m/s
z co-ordinate

Subscripts

c core
cl centreline
p pipe
r radius or rough
s smooth
0 at outside radius

Greek letters

α thermal diffusivity, $k/\rho\text{-}c_p$, m^2/s
β coefficient of volumetric thermal expansion, K^{-1}
Δ differential
δ boundary layer thickness, m
ε surface roughness, m
θ angle, $\circ$
μ dynamic viscosity, kg/m-s
v kinematic viscosity, m^2/s
ρ density, kg/m^3
τ shear stress, N/m^2

Introduction

When a fluid flows radially between two approximately parallel surfaces or discs a radial pressure gradient exists. If there is a temperature difference between the fluid and the surfaces, heat will be transferred. A practical example where such flow would occur is in the solar collector of a solar chimney power plant as shown schematically in Figure 1. This plant consists of a central chimney which is surrounded by a circular collector having a glass roof. Air is heated in the collector and flows up the chimney due to buoyancy effects. The air stream drives a turbine that is located near the base of the tower.

To evaluate the performance of such a plant the pressure drop and the heat transfer in the collector must be determined.

Analysis

Consider an elementary control volume in the boundary layer on a (smooth or rough) surface in a radial flow field between two discs (see Figure 2). The momentum equation in the radial direction is

$$\int_0^\delta p\,(r+\Delta r)\,\Delta\theta dz - \int_0^\delta \left(p + \frac{\partial p}{\partial r}\Delta r\right)$$

$$\times\,(r+\Delta r)\,\Delta\theta dz + \tau r\Delta\theta\Delta r$$

$$= \frac{\partial}{\partial r}\left(\int_0^\delta \rho v^2 r\Delta\theta dz\right)\Delta r - v_c\frac{\partial}{\partial r}\left(\int_0^\delta \rho v r\Delta\theta dz\right)\Delta r.$$

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For incompressible flow this reduces to

$$\frac{\partial}{\partial r}\left(\int_0^\delta v^2 r\,dz\right) - v_c\frac{\partial}{\partial r}\left(\int_0^\delta vr\,dz\right) + \frac{1}{\rho}\int_0^\delta r\frac{\partial}{\partial r}\,dz = \frac{\tau r}{\rho}. \tag{1}$$

In the core region outside the boundary layer the flow is essentially frictionless with the result that Bernoulli's equation is applicable:

$$p + \frac{\rho v_c^2}{2} = \text{constant}$$

or upon differentiation

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = -v_c\frac{\partial v_c}{\partial r}. \tag{2}$$

For fully developed turbulent flow in a pipe at $\mathrm{Re}_p \approx 10^5$ the velocity distribution can be approximated by a relatively simple empirical equation.[1] For developing flow over a flat surface the analogous velocity distribution (due to the universality of the velocity distribution in a turbulent boundary layer) may be expressed in terms of the core velocity v_c and the boundary layer thickness δ as

$$v = v_c\left(\frac{z}{\delta}\right)^{1/7}. \tag{3}$$

Substitute (2) and (3) into (1) and integrate, and find

$$-\frac{7}{72}rv_c^2\frac{d\delta}{dr} - \delta\left(\frac{7}{72}v_c^2 + \frac{23}{72}rv_c\frac{\partial v_c}{\partial r}\right) = \frac{\tau r}{\rho}. \tag{4}$$

For turbulent flow where the velocity distribution is relatively uniform at any radius (i.e. the boundary layer is relatively thin or the Reynolds number is high) the mass flow rate can be expressed approximately as

$$m \approx 2\pi r\,\rho v_c H$$

or

$$v_c \approx \frac{m}{2\pi r\rho H}. \tag{5}$$

To avoid excessively high radial velocities in the collector as the radius decreases, the height of the upper disc or collector roof is assumed to be given by

$$H = H_0\left(\frac{r}{r_0}\right)^{-b} \tag{6}$$

where b is a parameter ($0 \leq b \leq 1$).

Heat transfer coefficient: smooth surfaces

For smooth pipes the Fanning friction factor can be approximated by the following empirical equation[1] in the range $10^4 \leq \mathrm{Re}_p \leq 10^6$:

$$f = \frac{0.046}{\mathrm{Re}_p^{0.2}} = \frac{\tau_p}{0.5\rho v_p^2}$$

or

$$\tau_p = 0.023\frac{\rho v_p^2}{\mathrm{Re}_p^{0.2}}.$$

At a pipe Reynolds number of $\mathrm{Re}_p \approx 10^5$ the ratio of mean pipe velocity to centreline velocity is

$$\frac{v_p}{v_{pcl}} = 0.817$$

and hence

$$\tau_p = 0.01392\frac{\rho v_{pcl}^{1.8}}{(r_p/\nu)^{0.2}}.$$

The analogous relation applied to the boundary layer on a flat plate is

$$\tau = 0.01392\rho v_c^{1.8}\left(\frac{\nu}{\delta}\right)^{0.2}. \tag{7}$$

Substitute (5), (6), and (7) into (4), and find the differential equation

$$-r\delta^{0.2}\frac{d\delta}{dr} + \frac{16-23b}{7}\delta^{1.2} = 0.2068\left(\frac{\mu H_0 r_0^b}{m}\right)^{0.2} r^{1.2-0.2b}. \tag{8}$$

The solution of this differential equation together with the boundary condition $\delta(r_0) = 0$ (see Appendix A) yields the boundary layer thickness

$$\delta(r) = H_0\left\{\frac{1}{6.218-15.08b}\left(\frac{r_0}{H_0}\right)\left(\frac{\mu r_0}{m}\right)^{0.2}\left[\left(\frac{r}{r_0}\right)^{1.2-0.2b} - \left(\frac{r}{r_0}\right)^{2.743-3.943b}\right]\right\}^{5/6} \tag{9}$$

if $b \neq 0.4122$.

According to the Colburn analogy for a flat plate[1] the local heat transfer coefficient can be expressed in terms of the local shear stress:

$$\frac{h\,\mathrm{Pr}^{2/3}}{c_p\rho v_c} = \frac{\tau}{\rho v_c^2}. \tag{10}$$

From (5), (7), and (10) it then follows that

$$h = \frac{\tau c_p}{v_c\,\mathrm{Pr}^{0.667}} = 0.0032\frac{c_p}{\mathrm{Pr}^{0.667}}\left(\frac{m}{rH}\right)^{0.8}\left(\frac{\mu}{\delta}\right)^{0.2}$$

The local Nusselt number is defined as $\mathrm{Nu} = h(r_0 - r)/k$. From (6) and (9) it then follows for $b \neq 0.4122$

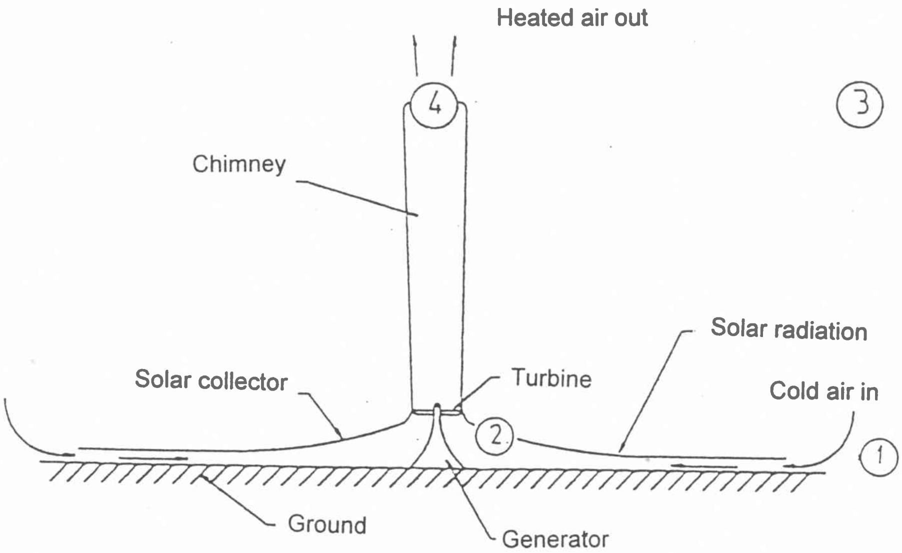

Figure 1 Solar chimney power plant

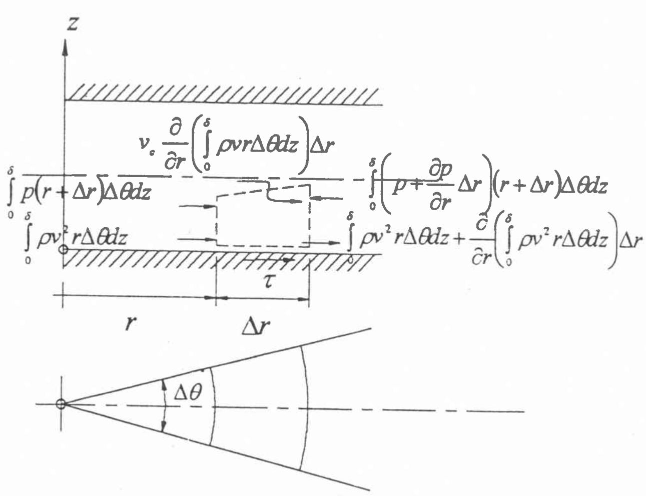

Figure 2 Boundary layer control volume

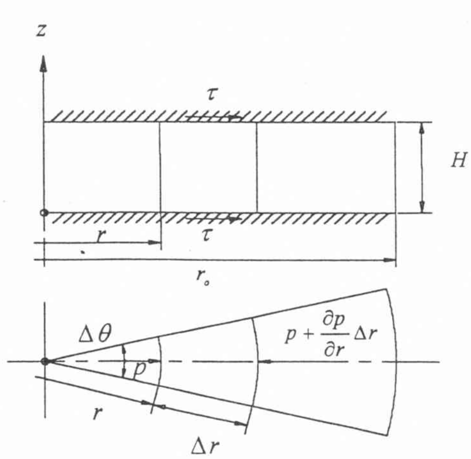

Figure 3 Control volume between two discs

that

$$\mathrm{Nu} = 0.0032\,\mathrm{Pr}^{0.333}\left(1-\frac{r}{r_0}\right)\left(\frac{m}{\mu H_0}\right)^{0.833}\left(\frac{r_0}{r}\right)^{0.8(1-b)} \times\left\{\frac{6.218-15.08b}{\left(\frac{r}{r_0}\right)^{1.2-0.2b}-\left(\frac{r}{r_0}\right)^{2.743-3.943b}}\right\} \tag{11}$$

The thermophysical properties in this equation are evaluated at the arithmetic average temperature of the surface and the core.

Heat transfer coefficient: rough surfaces

For rough pipe surfaces ($\varepsilon > 0$) the turbulent friction factor can be approximated by

$$f = 0.02975\left(\frac{\varepsilon}{d_p}\right)^{0.254}\left[1.75\left(\frac{\mu}{\rho v_p \varepsilon}\right)^{0.51}+1\right].$$

This empirical relation approximately correlates the friction data for rough pipes in the range $0.0001 < \varepsilon/d_p < 0.01$ and makes possible an analytical solution of the problem. The corresponding pipe shear stress is

$$\tau_p = \frac{f\rho v_p^2}{2}.$$

With $v_p = 0.817 v_{pcl}$ as before we find the analogous shear stress for turbulent flow over a rough flat plate:

$$\tau = 0.008326\rho v_c^2\left(\frac{\varepsilon}{\delta}\right)^{0.254}\left[1.94\left(\frac{\mu}{\rho v_c \varepsilon}\right)^{0.51}+1\right]. \tag{12}$$

Substituting (5), (6), and (12) into (4), the differential equation for δ becomes

$$-r\delta^{0.254}\frac{d\delta}{dr}+\frac{16-23b}{7}\delta^{1.254} = 0.08564\varepsilon^{0.254} \times\left[4.953\left(\frac{\mu H_0 r_0^b}{\varepsilon m}\right)^{0.51} r^{1.51-0.51b}+r\right].$$

The solution of this equation with $\delta(r_0) = 0$ is (see Appendix A):

$$\delta(r) = H_0\left(\frac{\varepsilon}{H_0}\right)^{0.2026}\left(\frac{r_0}{H_0}\right)^{0.7974} \times\left\{q\frac{\left(\frac{r}{r_0}\right)^{1.51-0.51b}-\left(\frac{r}{r_0}\right)^{2.866-4.1206}}{2.550-6.787b} + \frac{\frac{r}{r_0}-\left(\frac{r}{r_0}\right)^{2.866-4.120b}}{17.38-38.37b}\right\}^{0.7974} \tag{13}$$

if $b \neq 0.3757$ and $b \neq 0.453$, where

$$q = \left(\frac{\mu H_0 r_0}{\varepsilon m}\right)$$

The local Nusselt number for $\varepsilon > 0$ is

$$\mathrm{Nu} = \frac{h(r_0-r)}{k} = \frac{\tau c_p(r_0-r)}{k v_c \mathrm{Pr}^{0.667}} = 0.001325\,\mathrm{Pr}^{0.333} \times\left(1-\frac{r}{r_0}\right)\left(\frac{m}{\mu H_0}\right)\left(\frac{\varepsilon}{r_0}\right)^{0.2026}\left(\frac{r_0}{r}\right)^{1-b} \times\frac{4.953q\left(\frac{r}{r_0}\right)^{0.51(1-b)}+1}{\left\{q\frac{\left(\frac{r}{r_0}\right)^{1.51-0.51b}-\left(\frac{r}{r_0}\right)^{2.866-4.120b}}{2.550-6.787b}+\frac{\frac{r}{r_0}-\left(\frac{r}{r_0}\right)^{2.866-4.120b}}{17.38-38.37b}\right\}^{0.2026}}. \tag{14}$$

Pressure drop: smooth surfaces

To find the radial pressure drop due to friction in the collector, consider a control volume located between two discs as shown in Figure 3. Then

$$pr\,\Delta\theta\,H-\left(p+\frac{\partial p}{\partial r}\Delta r\right)(r+\Delta r)$$

$$\Delta\theta\,H+p\Delta r\,\Delta\theta\,H = -2\tau\,\Delta r\,r\,\Delta\theta$$

or

$$\frac{\partial p}{\partial r} = \frac{2\tau}{H}.$$

Substituting (6), (7), and (9) into (15):

$$\frac{\partial p}{\partial r} = 1.018 \times 10^{-3} \left(\frac{\mu^2}{\rho H_0 r_0}\right)\left(\frac{m}{\mu H_0}\right)^{1.833} \times \frac{1}{r_0}\left\{\frac{6.218 - 15.08b}{\left(\frac{r}{r_0}\right)^{12-17b} - \left(\frac{r}{r_0}\right)^{13.54-20.74b}}\right\}^{1/6}$$

and integrating between the outer radius r_0 and any other radius r, we find the pressure drop

$$p_{r0} - p_r = 1.018 \times 10^{-3}\left(\frac{\mu^2}{\rho r_0 H_0}\right)\left(\frac{m}{\mu H_0}\right)^{1.833} Fs\left(\frac{r}{r_0}\right)$$

where $x = r/r_0$,and

$$\begin{aligned} F_s(x) &= \int_x^1 f_s(t)\,dt\,, \\ f_s(t) &= \left\{\frac{6.218 - 15.08b}{t^{12-178b} - t^{13.54-20.74b}}\right\}^{1/6} . \end{aligned} \tag{16}$$

The integral in (16) is improper at both the upper limit $t = 1$ and the lower limit $t = x$ when $x \to 0$. For computational purposes (see Appendix B) the integral can be written as

$$F_s(x) = \begin{cases} \frac{1.2}{(\gamma-1)} \int_0^{(x^{1-\gamma}-1)^{0.833}} v^{0.2} g_s \times (1 + v^{1.2})\,dv, & 0 \le b \le 0.353 \\ \frac{1.2}{(1-\gamma)} \int_0^{(1-x^{1-\gamma})^{0.833}} v^{0.2} g_s \times (1 - v^{1.2})\,dv, & 0.353 < b \le 1 \end{cases} \tag{17}$$

where

$$\gamma = \begin{cases} 2 - 2.833b, & 0 \le b < 0.4122 \\ 2.257 - 3.457b, & 0.4122 < b \le 1 \end{cases}$$

and

$$g_s(s) = s^{\gamma/(1-\gamma)} f_s\left(s^{1/(1-\gamma)}\right) .$$

The integrals in (17) both have integrands with finite limits at both endpoints and can be integrated to a high degree of accuracy with a 4-point Gaussian integration rule. Alternatively, the integral can be approximated with an error of at most 5% in the range $0.6 \le x \le 1$, $0 \le b \le 1$ by

$$F_s(x) = (1-x)^{5/6}\left[1.51 + (1.71 - 2.5b)(1-x)^{5/6}\right] .$$

The loss coefficient based on inlet velocity due to friction on the two smooth surfaces in the collector is

$$\begin{aligned} K_0 &= \frac{p_{r_0} - p_r}{0.5\rho v_0^2} \\ &= 0.08038\left(\frac{\mu H_0}{m}\right)^{0.1667}\left(\frac{r_0}{H_0}\right) F_s\left(\frac{r}{r_0}\right) . \end{aligned} \tag{18}$$

Pressure drop: rough surfaces

Assuming the same roughness on both surfaces, and substituting (6), (12) and (13) into (15), find

$$\frac{\partial p}{\partial r} = 4.218 \times 10^{-4}\left(\frac{m^2}{\rho r_0 H_0^3}\right)\left(\frac{\varepsilon}{r_0}\right)^{0.2026}\frac{1}{r_0} f_r\left(\frac{r}{r_0}\right)$$

where

$$f_r(t) = \frac{4.953 q t^{0.51-0.51b} + 1}{\left\{ q\frac{t^{11.38-15.32b} - t^{12.74-18.93b}}{2.550 - 6.787b} + \frac{t^{10.87-14.81b} - t^{12.74-18.93b}}{17.38 - 38.37b}\right\}^{0.2026}} .$$

with q as before, and $t = r/r_0$. Integrating, we find

$$p_{r_0} - p_r = 4.218 \times 10^{-4}\left(\frac{m^2}{\rho r_0 H_0^3}\right)\left(\frac{\varepsilon}{r_0}\right)^{0.2026} Fr\left(\frac{r}{r_0}\right)$$

where

$$F_r(x) = \int_x^1 f_r(t)\,dt\,.$$

The integral is again improper at both limits, and may be written (see Appendix B)

$$F_r(x) = \begin{cases} \frac{1.254}{(\gamma-1)} \int_0^{(x^{1-\gamma}-1)^{0.7974}} v^{0.254} g_r \times (1 + v^{1.254})\,dv, & 0 \le b < 0.4 \\ \frac{1.254}{(1-\gamma)} \int_0^{(1-x^{1-\gamma})^{0.7974}} v^{0.254} g_r \times (1 - v^{1.254})\,dv, & 0.4 < b \le 1 \end{cases}$$

where

$$\gamma = \begin{cases} 2.202 - 3b, & 0 \le b < 0.454 \\ 2.581 - 3.834b, & 0.454 < b \le 1 \end{cases}$$

and

$$g_r(s) = s^{\gamma/(\gamma-1)} f_r\left(s^{1/(1-\gamma)}\right) .$$

For $0.6 \leq x \leq 1$ and $0.001 \leq q \leq 0.1$, the expression

$$F_r(x) = (1-x)^{0.797} \times$$

$$\left[1.97 + (2.1 - 3.21b + 4.79q - 9.18bq)(1-x)^{0.797}\right]$$

approximates the integral with an error of at most 10% in the range $0 \leq b \leq 1$, and less than 2% for $0.4 \leq b \leq 0.8$.

The loss coefficient for rough surfaces is

$$K_0 = \frac{p_{r_0} - p_r}{0.5\rho v_0^2} = 0.0333 \left(\frac{r_0}{H_0}\right)\left(\frac{\varepsilon}{r_0}\right)^{0.2026} F_r\left(\frac{r}{r_0}\right) . \quad (19)$$

Results

The development of the boundary layer during steady radial flow between two surfaces has been analysed. The results of the analysis are employed to predict the change in radial pressure due to frictional effects and the local heat transfer coefficient at the inlet of the collector of a solar chimney power plant.

For smooth surfaces the boundary layer thickness as given by equation (9) is shown graphically in Figure 4 for given values of b, r_0/H_0 and $\mu r_0/m$. Generally the flow between the surfaces becomes fully developed ($\delta \approx H_0/2$) fairly rapidly, whereafter the present analysis is no longer applicable. The boundary layer growth rate for rough surfaces as given by (13) is even more rapid than that for smooth surfaces. For the conditions specified, δ/H_0 is almost independent of the air mass flow rate, as shown in Figure 5.

The corresponding Nusselt numbers as given by (11) and (14) are shown in Figures 6 and 7, respectively. In both cases the Nusselt number decreases with increasing radius but improves with increasing air flow rate. A rough surface results in a higher heat transfer coefficient.

The corresponding loss coefficients are given by (18) and (19), respectively and are presented graphically in Figures 8 and 9.

Conclusions

Analytical solutions for predicting the pressure differential due to frictional effects and the heat transfer coefficient during developing radial flow between two essentially parallel discs are obtained. These results can be applied in the performance evaluation of a solar chimney power plant collector.

It should be noted that this analysis does not necessarily entirely model the true nature of the developing flow in the solar collector. Since the Rayleigh number $\mathrm{Ra}_H \gg 1708$ in this region, secondary flow patterns that affect the pressure drop and tend to augment the heat transfer may be present. When $\mathrm{Ra}_H > 10^8$ the Nusselt number due to these meandering counterflowing thermals $\left(\mathrm{Nu}_H \propto \mathrm{Ra}_H^{1/3}\right)$ becomes independent of the height of the collector roof above the ground.[2] Under certain operating conditions in certain areas of the collector the heat transfer due to the secondary flow may be of the same order or greater than that predicted by the present analysis.

References

1. Rohsenow WM & Choi HY. *Heat, Mass and Momentum Transfer.* Prentice Hall, Englewood Cliffs, New Jersey, 1961.

2. Bejan A. *Heat Transfer.* John Wiley and Sons, New York, 1993.

Appendix A

The solution of the differential equation

$$-r\delta^{\alpha-1}\frac{d\delta}{dr} + k\delta^{\alpha} = pr^{\beta} + qr$$

may be found as follows. The substitution $v = \delta^{\alpha}$ reduces the equation to the linear equation

$$-r\frac{dv}{dr} + \alpha k v = \alpha\left(pr^{\beta} + qr\right) .$$

The solution of the corresponding homogeneous equation is $v_h(r) = Ar^{\alpha k}$; assuming a particular solution of the form $v(r) = u(r)r^{\alpha k}$ and incorporating the boundary condition $\delta(r_0) = 0$, the solution is found as

$$\delta(r) = \left\{\frac{\alpha p}{\alpha k - \beta} r_0^{\beta}\left[\left(\frac{r}{r_0}\right)^{\beta} - \left(\frac{r}{r_0}\right)^{\alpha k}\right] + \frac{\alpha k}{\alpha k - 1} r_0 \left[\frac{r}{r_0} - \left(\frac{r}{r_0}\right)^{\alpha k}\right]\right\}^{1/\alpha}$$

if $\alpha k \neq \beta$ and $\alpha k \neq 1$; if $\alpha k = \beta$ the solution is

$$\delta(r) = \left\{\alpha p r^{\alpha k}\ln\frac{r_0}{r} + \frac{\alpha q}{\alpha k - 1} r_0\left[\frac{r}{r_0} - \left(\frac{r}{r_0}\right)^{\alpha k}\right]\right\}^{1/\alpha}$$

and if $\alpha k = 1$ then

$$\delta(r) = \left\{\begin{array}{l}\frac{\alpha p}{\alpha k - \beta} r_0^{\beta}\left[\left(\frac{r}{r_0}\right)^{\beta} - \left(\frac{r}{r_0}\right)^{\alpha k}\right] \\ + \alpha q r^{\alpha k}\ln\frac{r_0}{r}\end{array}\right\}^{1/\alpha} .$$

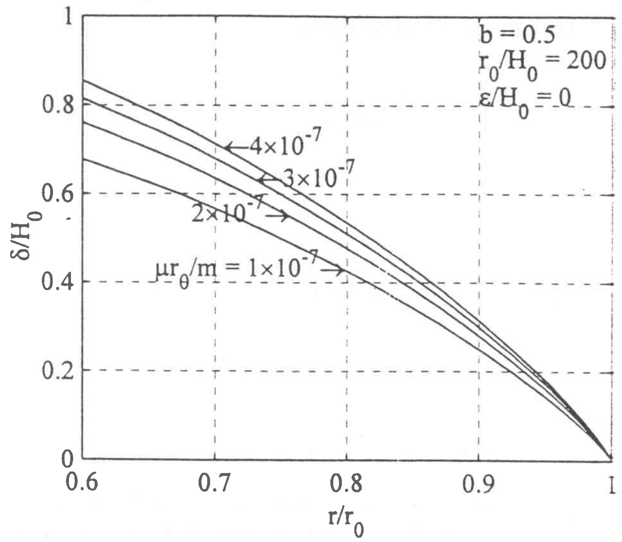

Figure 4 Boundary layer thickness (smooth surface)

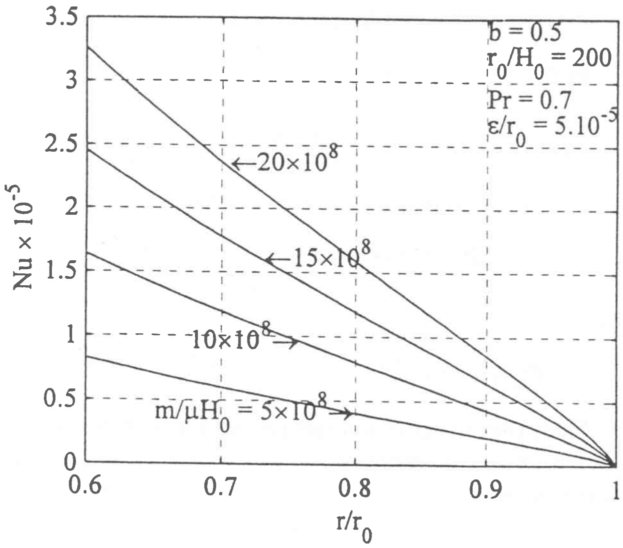

Figure 7 Nusselt number (rough surface)

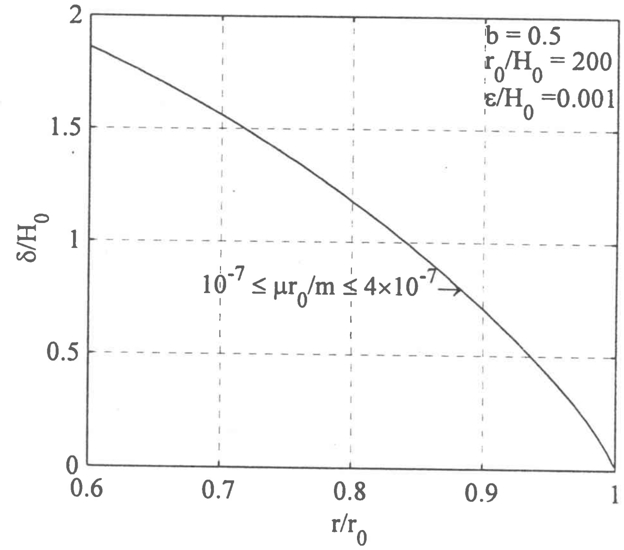

Figure 5 Boundary layer thickness (rough surface)

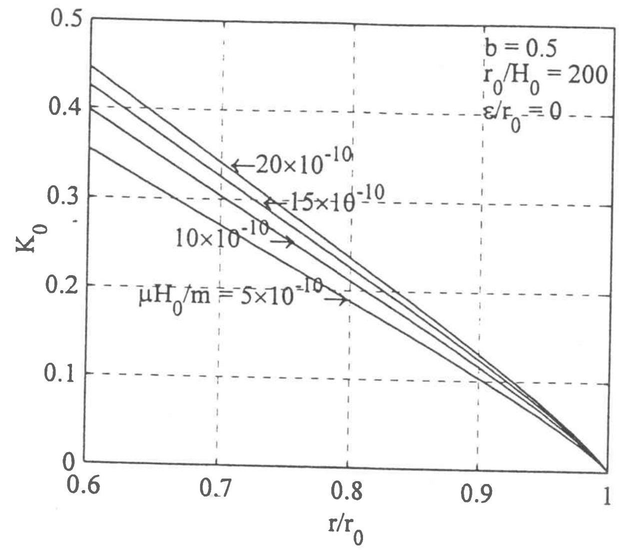

Figure 8 Pressure loss coefficient (smooth surface)

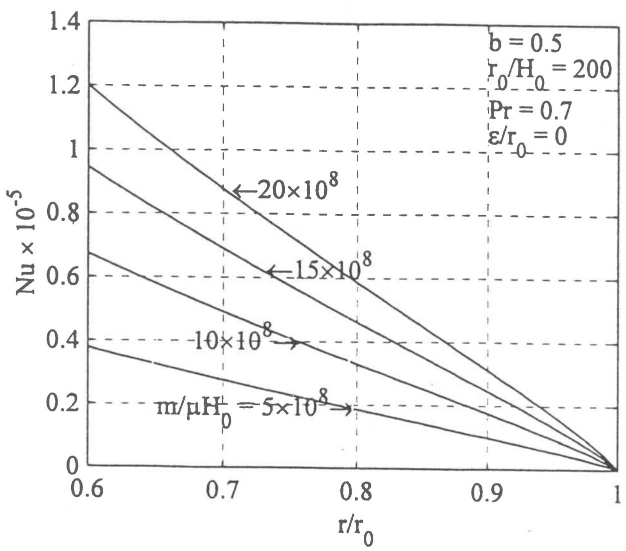

Figure 6 Nusselt number (smooth surface)

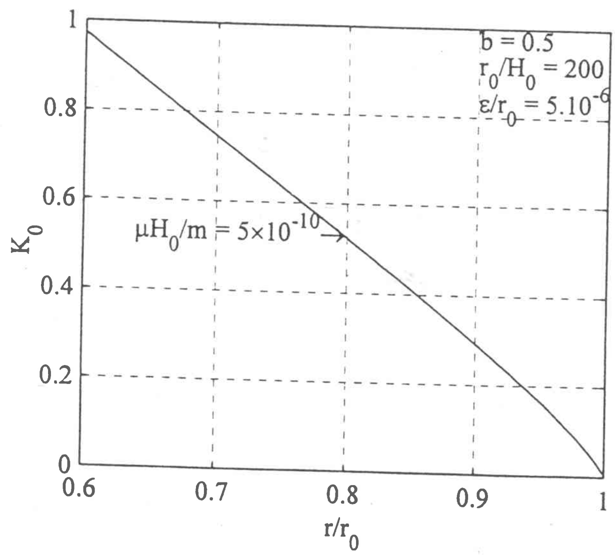

Figure 9 Pressure loss coefficient (rough surface)

Appendix B

Consider the integral

$$F(x) = \int_x^1 f(t)\,dt \quad (0 < x \le 1)$$

and suppose that γ and Γ are constants such that

$$\lim_{t\to 0+} t^\gamma f(t) = A \quad (A \neq 0,\ \gamma > 0)$$

and

$$\lim_{t\to 1-} (1-t)^\Gamma f(t) = B \quad (B \neq 0,\ 0 < \Gamma < 1).$$

Then the integral is improper and integrable at $t = 1$, and $F(x) \to \infty$ when $x \to 0$ if $\gamma > 1$. The successive substitutions

$$t = s^{1/(1-\gamma)}, \quad s = \begin{cases} 1 - v^{1/(1-\Gamma)}, & \gamma < 1 \\ 1 + v^{1/(1-\Gamma)}, & \gamma > 1 \end{cases}$$

reduce the integral to the form

$$F(x) = \begin{cases} \frac{1}{(1-\gamma)(1-\Gamma)} \int_0^{\left(1-x^{1-\gamma}\right)^{1-\Gamma}} v^{\Gamma/(1-\Gamma)} g \\ \quad \times \left(1 - v^{1/(1-\Gamma)}\right) dv, \qquad \gamma < 1 \\ \frac{1}{(\gamma-1)(1-\Gamma)} \int_0^{\left(x^{1-\gamma}-1\right)^{1-\Gamma}} v^{\Gamma/(1-\Gamma)} g \\ \quad \times \left(1 + v^{1/(1-\Gamma)}\right) dv, \qquad \gamma > 1 \end{cases}$$

where

$$g(s) = s^{\gamma/(1-\gamma)} f\left(s^{1/(1-\gamma)}\right).$$

These last integrals have integrands with finite limits at both upper and lower end-points of the integration interval.