Analytical model improvement using experimental modal results

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(Received June 1990)

Abstract

A method for improving dynamic finite element models using incomplete experimental modal models has been applied to a simple structure. This method uses the equations of motion and applies the Moore-Penrose generalised inverse to manipulate the rank deficient matrices involved. A simple prismatic beam structure was tested to assess the method. The experimental mode shapes were expanded to the full set of finite element model degrees of freedom. Translational degrees of freedom in the expanded model, which were not measured, were judged to be accurate. The expanded modes were then used to improve the finite element model's mass and stiffness matrices. The improved and original models were used to predict the effect of a physical modification to the struture. The improved model was seen to produce the more accurate prediction.

Nomenclature

The symbols used in this paper are similar to those used by O'Callahan [1]:

- E_n Expanded experimental mode shape matrix (n degrees of freedom)
- E_a Original experimental mode shape matrix (a "active" degrees of freedom)
- K_n Original FEM stiffness matrix
- K_n^I Improved FEM stiffness matrix
- M_n Original FEM mass matrix
- M_n^I Improved FEM mass matrix
- M_W Weighted change to original mass matrix
- U_n FEM mode shape matrix
- U_a Reduced FEM mode shape matrix
- X_n Physical displacement vector
- X_a Reduced physical displacement vector
- ω Frequency
- † Denotes the Moore-Penrose Generalised Inverse

Introduction

The Finite Element Method (FEM) is a numerical technique which is finding increasing application in many fields of engineering. However, in linear structural dynamics, inaccurate modelling of boundary conditions, joints and damping may severely limit the accuracy of the results. Therefore, before a FE model can form a basis for the purpose of predicting the effects of design changes or modifications to an existing structure, it is necessary to validate the model. An experimental modal analysis (EMA) of the structure or a prototype is usually conducted to provide this validation. If the correlation between the two models is found to be unsatisfactory for the purpose intended, it is necessary to improve the FE model. If experimental data (natural frequencies and mode shapes) is available, it would be advantageous to incorporate this information into the improvement technique.

Berman and Flannely [2] presented a method which utilises orthogonality conditions, the equations of motion and the theory of the Moore-Penrose generalised inverse to improve the mass and stiffness matrices of an analytical model using an incomplete experimental modal model. In recent years O'Callahan et. al. [1,3] have researched this method and included a modal expansion process for expanding the experimental mode shapes to the number of degrees of freedom used in the FE model. In this paper this method is described and applied within the SDRC I-DEAS software package and used to obtain an improved FE model which was used to investigate structural dynamic modifications.

Theory

Mode Shape Expansion

In order to apply the experimental mode shapes directly to improve the FE model's mass and stiffness matrices it is necessary that they should contain the same number of degrees of freedom as the FE model. Typically the experimental mode shape will contain fewer degrees of freedom than the FE model as it is not always possible to test all the points included in the FE model and because rotational accelerations are not usually measured experimentally. In the past methods such as Guyan reduction [4] have been used to reduce the FE model to the test degrees of freedom. This however is a static reduction and therefore produces errors in dynamic analysis, especially when applied to a structure with fairly evenly distributed mass. An exact reduction was used by Kammer [5], the basis of which is the same as in the expansion used by O'Callahan [1,3].

The expansion technique used by O'Callahan applies the Moore-Penrose generalised inverse which is described briefly in Appendix A. He employed the generalised inverse to calculate a *unique* transformation which maps the

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set of reduced FE mode shapes, containing only the test degrees of freedom, to the FE mode mode shapes containing the full set of analytical degrees of freedom. This unique mapping is then applied to the experimental mode shapes to produce the expanded experimental mode shapes (containing the full set of analytical degrees of freedom). The procedure is outlined below:

The mode shapes (eigenvectors) represent a transformation from physical to modal space.

$$X_n = U_n P \tag{1}$$

P is a column vector of the generalised co-ordinates in modal space, with dimension equal to the number of modes included in mode shape matrix U_n. Therefore, considering the reduced model (the same number of modes but fewer degrees of freedom), the equivalent expression,

$$X_a = U_a P \tag{2}$$

contains the same vector P. Because the mode shape matrices are not square it is necessary to use the Moore-Penrose generalised inverse to formulate a mapping between the full and reduced displacement vectors. Writing equation (2) as $P = U_a^{\dagger} X_a$ and substituting it into equation (1) gives:

$$X_n = U_n U_a^{\dagger} X_a \tag{3}$$

or

$$X_n = T_u X_a \tag{4}$$

where T_u is the transformation matrix required to map the experimental mode shapes to the expanded set of experimental mode shapes as follows:

$$E_n = T_u E_a \tag{5}$$

FE Mass Matrix Improvement

The mass matrix improvement process involves modifying the original mass matrix M_n, and formulating the improved mass matrix M_n^I, so as to satisfy the condition.

$$E_n^T M_n^I E_n = I \tag{6}$$

Where E_n is the expanded mass normalised experimental mode shape matrix.

The final improved mass matrix, M_n^I is composed of the original mass matrix as obtained from the FE model, combined with a matrix M_w, constituting the difference between the two:

$$M_w = M_n^I - M_n$$

Since matrix M_w is not unique, and as the FEM model should be sufficiently accurate to describe the general mass distribution in the structure, the approach is to find the mass change matrix M_w such that it requires the minimum change to the original FEM mass matrix M_n to satisfy the orthogonality condition described in equation (6).

Thus it is required to minimise the Euclidean norm of:

$$M_W = N_A^{-I}(M_n^I - M_n)N_A^{-I}. \tag{7}$$

Which implies the minimisation of the change to the original mass matrix with the weighting factors (N_A^{-I}) applied. These weighting factors may be chosen to sensitise the minimisation procedure such that certain regions of the model are most affected. Alternatively, if the FEM mass matrix is considered to be representative of the mass distribution, a more uniform difference would exist. In this case, it would be advantageous to normalise the matrices such that the algorithm's sensitivity is proportional to the magnitudes of the original FEM mass matrix elements. This may be accomplished by selecting N_A^{-I}, such that $N_A^2 = M_n$. Substituting M_N^I from equation (7) into equation (6) gives:

$$E_n^T(N_A M_W N_A + M_n)E_n = I \tag{8}$$

which can be rewritten in the form of equation (A.1) in Appendix A.

$$E_n^T N_A M_W N_A E_N = I - E_n^T M_n E_n \tag{9}$$

From Appendix A the minimum-norm least-squares solution to the above equation is,

$$M_W = (E_n^T N_A)^{\dagger}(I - E_n^T M_n E_n)(N_A E_n)^{\dagger} \tag{10}$$

Construction of the generalised inverses (see Appendix A.3) and application of equation (7) gives the required expression for the improved mass matrix.

$$M_n^I = M_n + M_n E_n (E_n^T M_n E_n)^{-I} (I - E_n^T M_n E_n)(E_n^T M_n E_n)^{-I} E_n^T M_n \tag{11}$$

FE Stiffness Matrix Improvement

The formulation of the stiffness matrix improvement used in this paper differs from that used by O'Callahan [3], but produces very similar results. The improved stiffness matrix must satisfy the equations of motion $K_n^I E_n = \omega^2 M_n^I E_n$. Premultiplication by E_n^T gives:

$$E_n^T K_n^I E_n = \omega^2 \tag{12}$$

Following the same procedure as was used for the mass matrix improvement but using $N_A^2 = M_n^I$ gives the following result:

$$K_n^I = K_n + M_n^I E_n(\omega^2 - E_n^T K_n E_n)E_n^T M_n^I \tag{13}$$

Experimental model

Structure Dimensions

The test specimen was a small, lightly damped structure made of mild steel. The shape of this structure resembled an H, machined from a single piece of steel. Mortise joints were machined at the ends of the frame to allow future modification, and the assessment of joints. The dimensions and details of the structure are shown in Fig. 1.

Test Set-Up

The structure was suspended from two soft elastic cords at points A and B (see Fig. 1) which a previous FE analysis had shown to be nodes in the frequency range of interest. The excitation was applied by two electro-dynamic shakers attached in orthogonal directions at the points C and D shown in Fig. 1. The structure was excited by two *uncorrelated* random signals over the frequency range of interest, as required for multiple exciter testing. The accelerations were measured at 38 points on the structure in the two directions that were excited. The accelerometers used were PCB Structcells. The outputs of the two force transducers and the two accelerometers were analysed with a Genrad 2515 to produce the required frequency response functions.

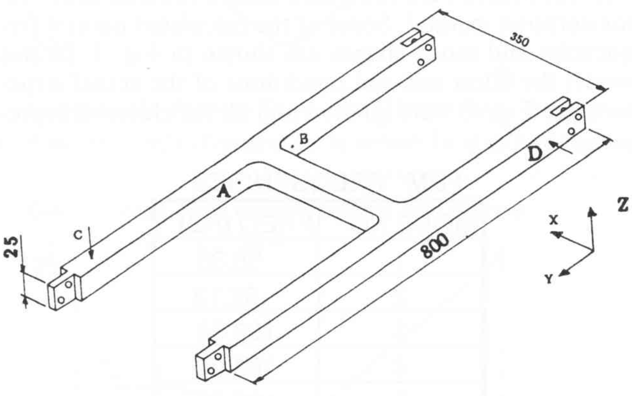


Figure 1 – The Test Structure

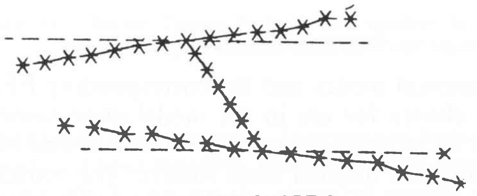
Modal Model

The SDRC I-DEAS package extracted the modal model from the frequency response functions using the Polyreference time domain modal extraction technique. The natural frequencies and some of the mode shapes are shown in Fig. 2. The mode shapes are poor considering the simplicity of the structure tested. It is believed that this is due to the effect of the mass of the accelerometers being moved around the lightly damped structure, since the polyreference technique is more suited to structures with viscous damping and is very sensitive to changes in the structure during testing. This is substantiated by the fact that previous tests of the structure with an impact hammer produced smooth mode shapes (the accelerometers remain stationery while the direction of the impact and its point of application change during the test). These mode shapes however also showed irregularities at the accelerometer locations. Nevertheless it was decided to use the relatively poor mode shapes in the expansion and model improvement process as this would test the capability of the expansion process to smooth irregular mode shapes.

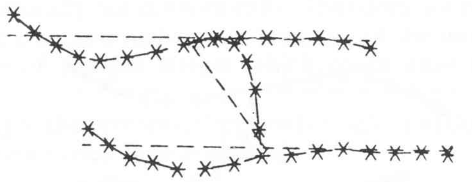
The structure was retested with an impact hammer to obtain the best possible modal model. This modal model was employed for the modification prediction, whilst the former was employed to assess the expansion reduction and hence smoothing capabilities of the process.

EXPERIMENTAL FREQUENCIES

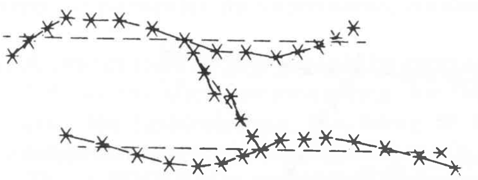
MODE NO	FREQ (HZ)
1	57.97
2	67.24
3	143.33
4	203.46
5	213.35
6	589.87
7	600.56
8	629.24



MODE SHAPE 2



MODE SHAPE 6



MODE SHAPE 7

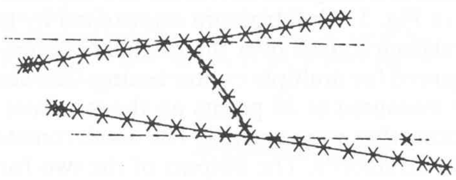
Figure 2 – Experimental Results

Analytical model

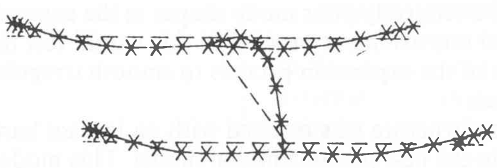
The structure was modelled using 46 nodes connected by linear Timeshenko beam elements and the first twenty free-free modes were computed using a simultaneous vector iteration method. Some of the calculated natural frequencies and mode shapes are shown in Fig. 3. In this model the fillets and end conditions of the actual structure (see Fig. 1) were ignored and all the elements represented a 25 × 25 mm section. The most obvious effect of this approximation was to produce the symmetric mode number 6 (Fig. 3). Modelling the end conditions more accurately did produce an unsymmetric mode shape very similar to the corresponding experimental mode shape (Fig. 2). The cruder model was chosen to investigate the model improvement process.

FEM FREQUENCIES

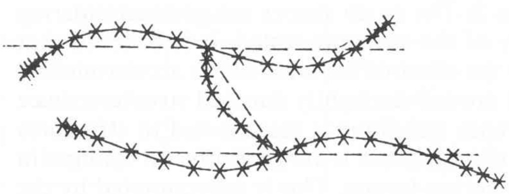
MODE NO	FREQ (HZ)
1	56.35
2	62.13
3	129.30
4	183.92
5	196.22
6	579.47
7	567.28
8	592.19



MODE SHAPE 2



MODE SHAPE 6

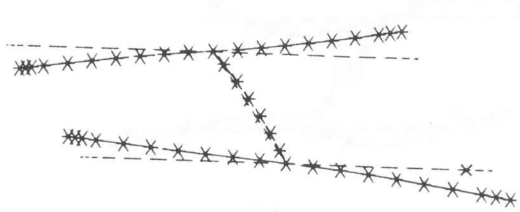


MODE SHAPE 7

Figure 3 – Original FEM Modal Results

Improved model

Eight experimental modes and the corresponding FEM modes were chosen for use in the model improvement process. The FEM mode shapes were scaled to unit modal mass using the original mass matrix. The reduced FEM mode shapes were then extracted from this matrix and the experimental mode shapes were scaled to the largest value in the reduced FEM mode shape. The I-DEAS software has a facility which allows matrices to be manipulated by a number of standard matrix operations. This facility was used to perform the matrix calculations required by the expansion and improvement processes. Some of the resulting smoothed mode shapes are shown in Fig. 4. It can be seen that the expansion process has not only expanded the mode shapes correctly but has also smoothed them completely. Mode shape 7 and to a lesser extent mode shape 2 show deflections of the cross member which were not shown in the original experimental mode shapes. Therefore these degrees of freedom which were not measured have been simulated by the expansion process. Unfortunately it is not yet possible to measure rotational motions for comparison with the calculated values.



MODE SHAPE 2

MODE SHAPE 7

Figure 4 – Expanded Mode Shapes

Appendix A: The Moore-Penrose Generalised Inverse

A.1 Introduction

This appendix presents a brief description of the theory of the generalised inverse. The interested reader requiring a more comprehensive mathematical description is referred to the text by Ben-Israel and Greville [6].

E. H. Moore introduced the subject in 1920, referring to it as "the reciprocal of the general algebraic matrix". In this paper he defined a unique inverse for all matrices. It was, however, only in the 1950's that the least-squares properties were discovered and extended by Bjerhammar and Penrose respectively [6].

A.2 Penrose Conditions

Moore showed that there exists a *unique* inverse for any general nonzero matrix. Penrose presented four conditions for the calculation of the unique inverse. "The four equations:

- $AXA = A$,
- $XAX = X$,
- $(AX)^* = AX$,
- $(XA)^* = XA$

have a unique solution for any A." – Penrose [7].

This unique solution is often denoted by $A^\dagger$ and is known as the Moore-Penrose generalised inverse. Other non-unique generalised inverses which do not satisfy all of the above four conditions can be constructed. These inverses have useful least-squares properties which will be mentioned later. The notation used for these inverses is that used in ref [6]. A generalised inverse satisfying the first and third conditions, for example, will be denoted as $A^{(1,3)}$ and is an element of A{1,3} the set of all generalised inverses satisfying the first and third Penrose conditions.

A.3 Construction of Generalised Inverses

As only the unique generalised inverse $A^\dagger$ is used in this paper the construction of non-unique inverses [6] is not presented.

If matrix A is a m × n matrix of rank r, ($r > 0$) then there exists a factorisation $A = FG$ such that F is a m × r matrix of rank r and G is a r × n matrix also of rank r. This is known as the full-rank factorisation of matrix A. MacDuffee showed that the full-rank factorisation is related to the Moore-Penrose generalised inverse by the following formula[6]:

$$A^\dagger = G^*(F^*AG^*)^{-1}F^*$$

Note that if A is a real matrix the Hermitian transpose (denoted by *) reduces to the normal transpose.

A.4 *Least-Squares and Minimum-Norm Properties*

If $Ax = b$ is an inconsistent linear system of equations, with A of size m × n and b of size m × 1, a residual vector $r = b - Ax$ can be defined. It is often desirable to find the solution x which will make the Euclidean norm of the residual vector a minimum. ie. To make $\Sigma_{i=1}^{n} \mid r \mid^2 = \| b - Ax \|^2$ a minimum. It can be shown that this is achieved by using any generalised inverse of A which satisfies the first and third Penrose conditions. Because this is not a unique generalised inverse there are infinite solutions (x) which satisfy the imposed condition. These solutions may be written as:
$x = A^{(1,3)}b + (I - A^{(1,3)}A)y$,
where y is an arbitrary nx1 vector. Note that if A is of full column rank then the above expression produces a unique solution.

If there are an infinite number of solutions it is necessary to impose additional constraints on the solution in order to achieve a unique solution. For example, it is possible to find a minimum norm solution, (a solution which makes $\| x \|^2$ a minimum). It can be shown that such a solution is found by $x = A^{(1,4)}b$.

"The least-squares solutions of $Ax = b$ coincide with the solutions of $Ax = AA^{(1,3)} b$" [6]. The minimum-norm solution of the latter equation is therefore $x = A^{(1,4)}AA^{(1,3)} b$.

It was shown by Urquhart that $A^{(1,4)}AA^{(1,3)} = A^\dagger$, therefore $x = A^\dagger b$ is the minimum-norm least-squares solution to the inconsistent linear system of equations $Ax = b$.

Using this result it can be shown that the minimum-norm least-squares solution to the inconsistent matrix equation

$$AXB = D \qquad (A.1)$$

is, $X = A^\dagger DB^\dagger$.