A new definition for laminar flow entrance lengths of straight ducts

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Abstract

A new definition for hydrodynamic entrance lengths of developing laminar flow in straight ducts of arbitrary cross-section is proposed. The suggested length of flow development is theoretically based on hydrodynamic characteristics of the flow development and can be retrieved analytically whenever the fully developed velocity profile is known. A property is obtained which is both unique and physically more meaningful than the customary hydrodynamic entrance length. The latter corresponds to the maximum velocity attaining an arbitrary percentage of the fully developed limit, this value being obtainable only indirectly by numerical means. The proposed new definition forsakes detail of the velocity profile for the more fundamental equalisation of the two basic conditions of developing and developed flow. The definition is applied to some commonly encountered duct shapes and corresponding hydrodynamic entrance lengths are provided.

Nomenclature

a	physical constant, m
a	cross-sectional dimension of rectangular duct, m
b	cross-sectional dimension of rectangular duct, m
D	hydraulic diameter, m
f	axially local friction factor, $(D/(2\rho u_m^2))\left(\frac{dp}{dx}\right)$
f_{app}	apparent friction factor, $(D/(2\rho u_m^2))\left(\frac{\delta p}{\delta x}\right)$, with δp, δx differences between inlet and present position values
K	dimensionless incremental pressure drop number, $2\delta p/(\delta u_m^2) - fx/D$
L	hydrodynamic entrance length, m
m	power exponent
n	power exponent
p	pressure, $kg.m^{-1}.s^{-2}$
Re	Reynolds number, $\rho u_m D/\mu$
r	radius, m
r^*	dimensionless parameter, r_i/r_o
u	fluid velocity, $m.s.^{-1}$
u_m	mean value of u, $m.s.^{-1}$
x	axial distance m
x^+	dimensionless axial distance, $y/(DRe)$
y	generic variable
α^*	aspect ratio
μ	fluid dynamic viscosity, $kg.m^{-1}.s^{-1}$
ρ	fluid density, $kg.m^{-3}$
ϕ	half apex angle of isosceles triangle, degrees

Superscripts

$+$	dimensionless entity

Subscripts

c	critical point
i	inner radius
ic	based on inviscid core approximation
K	based on incremental pressure drop number
l	fully developed
m	mean value
max	maximum value
o	outer radius
0	inlet value
υ	based on maximum velocity

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Introduction

Present day engineering practice often requires a high degree of accuracy in the design and control of advanced flow processes. In many process considerations, it becomes necessary to include deviations from fully developed flow analyses caused by duct inlets and bends. In the case of relatively short ducts, the hydrodynamic entrance region, where the flow is developing hydrodynamically from a specified entrance profile (usually taken as uniform over the cross-section) to an almost fully developed profile, is of major importance as it is here that flow properties deviate most significantly from the fully developed state.

Immediately beyond the duct inlet, velocity overshoots may occur, the maximum local velocity varying with axial position both in magnitude and radial position. This is also referred to as the inflection phenomenon [1]. At the circumference along the duct wall, a developing boundary layer exists which increases in thickness until, at some position downstream, opposite sides merge and the boundary layer fills the entire duct. The distance measured from the duct inlet to this position is called the entry length after Mohanty and Asthana [2]. Downstream of this entry length, but still within the entrance region, the velocity profile is still developing although the boundary layers have merged. At some stage only very small, diminishing changes in velocity occur and the flow is considered to be hydrodynamically fully developed. The region from the inlet to this position is known as the entrance region and it extends over a certain distance, the entrance length. Figure 1 gives a schematic illustration of these two lengths.

The entrance region of laminar developing flow in a

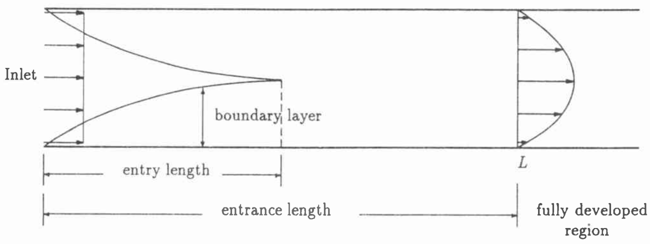

Figure 1 – Schematic layout for physical interpretation of entrance and entry lengths.

straight duct can be characterised by changes in several physical parameters. The apparent friction factor is the average friction between the inlet and the station under consideration, thus $f_{app} = \frac{1}{\delta x}\int_0^{\delta x} f\,dx$. Both these friction factors depend on the mean velocity gradient at the wall for the station cross-section and decrease monotonically in the streamwise direction. Another parameter which changes in the entrance region is the magnitude of the maximum velocity (centre-line velocity in the case of regular duct shape). With the exception of possible velocity overshoots, it increases asymptotically from the uniform inlet value to its maximum value in the fully developed region.

The generally accepted specification of a hydrodynamic entrance length, which is determined by the position at which the maximum velocity attains 99% of the fully developed limit value, usually presents some difficulties when numerical prediction of such a flow is analysed. The arbitrary assignment of this 99% condition governing the length over which flow development is taking place, is highly unsatisfactory. In cases of unknown fully developed velocity profiles, the exact cross-sectional location of maximum velocity is unknown. Computation of the maximum velocity may then require a painstaking numerical process of trial and error computations, often coupled with grid alterations. Inconsistency in the literature as to numerical values of entrance lengths for various duct shapes is evidence of the unfortunate consequences of this definition.

Another, similarly arbitrary, definition of the entrance length is given by Fleming and Sparrow [3] as $L_K^+ = x^+$, with x^+ the position where the incremental pressure drop number $K(x^+)$ reaches 95% of the value of $K(\infty)$. They also define L_u^+ as that value of x^+ at which $u_{max}(x^+) - u_m$ reaches 95% of $u_{max}(\infty) - u_m$.

The intention here is to propose an appropriate definition for entrance length which has both consistency and direct physical meaning. The underlying assumption, based on the work of Churchill and Usagi [8], is that the method of asymptote matching leads to an expression which approximately describes "phenomena in which the dependent variable varies uniformly from small to large values of the independent variable." For laminar flow in a straight duct, the initial uniform velocity profile and the fully developed profile far downstream are considered to be the two asymptotic conditions of a spatially changing transitional velocity profile between the two limits. This model, with axial distance as the independent variable, is used to describe flow development. A definite physical transition point at the intersection of two asymptotes, which correspond respectively with the developing section and the fully developed section, exists. The position of this transition point is independent of the accuracy of any computational flow field calculations and therefore provides a natural criterion for the definition of hydrodynamic entrance length of developing flow in a straight duct.

In this paper, asymptotic functions for the friction factor-Reynolds number product are employed to derive a general condition for duct entry lengths. The choice of this product as the dependent variable is based upon the friction factor having greater significance in engineering practice than the maximum velocity value. The friction factor is defined in terms of the velocity gradients normal to the wall. These velocity gradients govern hydrodynamic as well as heat transfer phenomena, and a criterion based on such properties would have greater physical significance than one based on a percentage of maximum velocity. A definition based on the friction factor was thus felt to be more appropriate than one based on the maximum velocity value, which occurs somewhere near the middle of the duct.

A generalisation of flow development approximations for parallel plates and rectangular ducts, developed for flow through porous media by du Plessis and Masliyah [4,5] is presented here. A similar approach, using such a hybrid combination of asymptotic results, was successfully followed by the author [6] to model flow conditions through an empty cooling tower shell. Air flow through Borda mouths and slits were used as the asymptotic solutions to this problem.

Mathematical Analysis of Asymptote Matching

Let $y(x)$ be the representation of a physical property of laminar flow developing hydrodynamically in a straight duct from a uniform inlet velocity profile. The duct is assumed to be of constant and invariable cross-section. Let the limiting condition on $y(x)$, at the onset of flow development at the immediate entrance to the duct, be governed by a power dependence on x,

$$y(x \to 0) = y_0 = a_0 x^{m_0}. \tag{1}$$

In the fully developed region when x is large, $y(x)$ may be governed by a different power law dependence on x and accordingly it follows that

$$y(x \to \infty) = y_\infty = a_\infty x^{m_\infty}. \tag{2}$$

These two asymptotic conditions may now be matched by straight-forward addition, a procedure not uncommon in engineering practice [7], so that

$$y(x) = y_0 + y_\infty. \tag{3}$$

This result present a function which gives the correct asymptotic behaviour of $y(x)$ although the intermediate part may be totally unrealistic, as y_0 dominates the equation as $x \to 0$ and y_∞ dominates as $x \to \infty$. A remarkable improvement in correlation with experimental results can be obtained by shifting the intermediate part of $y(x)$

through the raising of each term of equation (3) to some power n as follows:

$$y(x)^n = y_0^n + y_\infty^n. \tag{4}$$

The effect of an increase in n is to shift the solution,

$$y_n \equiv y(x) = (y_0^n + y_\infty^n)^{1/n}, \tag{5}$$

nearer to the asymptotes. This shift is most pronounced in the region close to the critical point x_c where the two asymptotes (1) and (2) meet. This critical point, with a corresponding critical value of y_c, is given by the x-value which satisfies the condition

$$y_0 = y_\infty. \tag{6}$$

The function $y(x)$ may be normalised with respect to either of the asymptotes as follows:

$$y/y_0 = [1 + (y_\infty/y_0)^n]^{1/n} \tag{7}$$

$$\text{or} \quad y/y_\infty = [1 + (y_0/y_\infty)^n]^{1/n}. \tag{8}$$

Although the choice of normalised equation has no quantitative influence on the final results, the latter normalisation will be used in this work.

Apparent Friction Factor

According to Shah and London [9], it is clear that the apparent friction factor-Reynolds number product for laminar uniform entry flow in a very short duct of arbitrary cross-section, is given by

$$f_{app}Re = \frac{3{,}44}{\sqrt{x^+}} \tag{9}$$

as $x \to 0$. This is also the result which holds for flow development along a flat plate and should be so, since, for a very short duct the flow is governed by local conditions at the surfaces, regardless of the cross-sectional shape of the duct.

At the other extremity, when the flow is fully developed within a long duct, the apparent friction factor tends to the constant friction factor for fully developed flow, so that as $x \to \infty$,

$$f_{app}Re = f\,Re. \tag{10}$$

Addition of the asymptotic conditions according to equation (5) then yields

$$f_{app}Re = \left[(f\,Re)^n + \left[\frac{3{,}44}{\sqrt{x^+}} \right]^n \right]^{1/n}. \tag{11}$$

According to equation (8) this equation can be normalised as follows

$$\frac{f_{app}Re}{f\,Re} = \left[1 + \left[\frac{3{,}44}{f\,Re\sqrt{x^+}} \right]^n \right]^{1/n}. \tag{12}$$

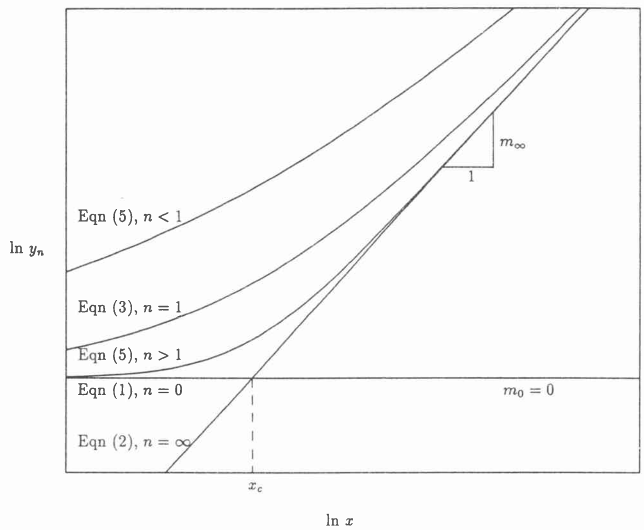

Figure 2 – Illustration of mathematical analysis, equations (1), (2), (3) and (5).

This normalised expression is graphically illustrated in Figure 2 for different values of n. It is clear that n is a shifting parameter which governs the rate of the transition between the developing and the fully developed regions. An increase in n causes the transition to be more abrupt and the function to follow the asymptotes more closely.

The hydrodynamic developing length for laminar flow developing from a uniform inlet profile in a straight duct, is now defined as the distance between the inlet and the critical point x_c^+, where the asymptotes intersect. This leads to the non-dimensional condition:

$$L_c^+ = x_c^+ = \left[\frac{3{,}44}{f\,Re} \right]^2. \tag{13}$$

It is worthwhile noting that this definition of the hydrodynamic flow developing length is also independent of the exponent n. The possible calculation procedures for n are covered extensively in the paper by Churchill and Usagi [8].

Substitution of equation (13) in equation (12) leads to the useful expression

$$f_{app}Re = f\,Re \left[1 + \left(\frac{L_c^+}{x^+} \right)^{n/2} \right]^{1/n}. \tag{14}$$

Equation (14), which is applicable to a duct of arbitrary cross-section, is graphically presented in Figure 3 for some values of n. The $(f_{app}Re$ value at the transition point where $x^+ = L_c^+$ is then given by the following expression

$$(f_{app}Re)_c = 2^{1/n} f\,Re. \tag{15}$$

In the fortunate case where the numerical value of $(f_{app}Re$ is known at the transition point, the value of n could be obtained directly from the following equation

$$n = \frac{\ln 2}{\ln \left[\frac{(f_{app}Re)_c}{f\,Re} \right]}. \tag{16}$$

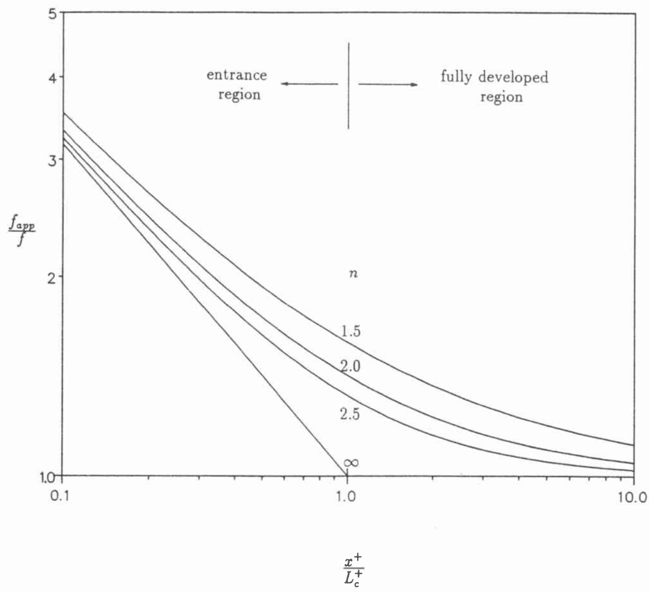

Figure 3 – Apparent friction factor for hydrodynamically developing duct flow.

Applications

Circular Tube

The hydrodynamic entrance region of a straight tube is a seemingly simple phenomenon which has been the subject of numerous studies in the past. The extent of the hydrodynamic entrance region has, however, as yet not been resolved satisfactorily, because of the mentioned awkwardness imbedded in the various percentage criteria for its definition.

According to equation (13) the developing length for a straight tube with Fanning friction factor, $f\,Re = 16$, is given by

$$L_c^+ = 0{,}0462. \quad (17)$$

Table 1 gives values obtained for L_c^+, calculated according to (13), together with published results for L_{ic}^+ from McComas [11] and L_K^+ and L_u^+ from Fleming and Sparrow [3]. The data does not demonstrate an equivalence in numerical values, but rather gives an indication that the present definition yields comparable results. According to Shah and London [9] (their equation (192)), the value for $f_{app}Re$ at the developing length $L_c^+ = 0{,}0462$ is 22,01, which leads to an n-value of 2,17 according to equation (16). These results are contained in Table 2 as the limiting case for concentric annular ducts with zero inner radius.

Table 1: Comparison of entrance lengths for different ducts and different definitions.

L_{ic}^+ – McComas [11] Tables 1, 3 and 4.

L_K^+, L_u^+, – Fleming and Sparrow [3] Table 2.

– Fleming and Sparrow [3] Table 2, equation (23) and Table 3 below.

Configuration	L_{ic}^+	L_K^+	L_u^+	L_c^+
parallel plates ($r^*=1$)	0,00588	0,0083	0,0090	**0,0205**
5:1 rectangular	0,0131#	0,030	0,052	0,0325
circular ($r^* = 0$)	0,0260	0,038	0,044	**0,0462**
2:1 rectangular	0,0255	0,0047	0,070	0,0490
square	0,0328			0,0585
equilateral triangular	0,0398	0,071	0,080	0,0666
30° isosceles triangular	0,0435	0,096	0,103	0,0693

Table 2: Concentric annular duct flows, where r^*-values of 0 and 1 correspond respectively to a circular tube and parallel plates. $f\,Re$ – McComas [11] Table 1, Shah and London [9] Table 93.

r^*	$f\,Re$	L_c^+	$(f_{app}Re)_c$	n
0,00	16,00	**0,0462**	22,0	2,17
0,05	21,57	0,0254	29,6	2,19
0,10	22,34	0,0237	30,5	2,27
0,50	23,81	0,0209	32,0	2,35
0,75	23,97	0,0206	32,1	2,37
1,00	24,00	**0,0205**	32,1	2,38

Equation (14) now yields the following expression for the friction factor-Reynolds number product

$$f_{app}Re = 16\,[1 + (0{,}046/x^+)^{1,1}]^{0,46}. \quad (18)$$

This expression correlates closely with numerical data and empirical expressions presented by Shah and London [9] for the problem under consideration.

Concentric annular ducts

In the case of concentric annular ducts, a non-dimensional parameter for the radii is defined as

$$r^* = r_i/r_o. \quad (19)$$

Application of the present results to hydrodynamically developing flow in a concentric annular duct leads to results presented in Table 2. Values for $f\,Re$ were taken from Shah and London [9] (their Table 93) and apparent friction factors were obtained from interpolation of data given by Shah and London [9] in their Table 93, while L_c^+ and n were calculated using equations (13) and (16) respectively.

Rectangular ducts

The aspect ratio for the rectangular cross-section shape of such a duct is defined as

$$a^* = 2b/2a, \quad (20)$$

where $2a$ and $2b$ denote the lengths of the longer and the shorter sides of the duct cross-section respectively. Derivations similar to that of tube flow lead to the results presented in Table 3. Again the $f\,Re$ values were obtained from Shah and London [9] (their Table 42) and equation (13) was used to obtain L_c^+. The apparent friction factors

Table 3: Rectangular duct flows, where α^*-values of 1 and 0 correspond respectively to a square duct and parallel plates. $f\,Re$ – Shah and London [9] Table 42.

α^*	$f\,Re$	L_c^+	$(f_{app}Re)_c$	n
1,000	14,22708	0,0585	20,10	2,01
0,9	14,26098	0,0582		
5/6	14,32808	0,0576		
0,800	14,37780	0,0572		
3/4	14,47570	0,0565		
0,700	14,60538	0,0555		
2/3	14,71184	0,0547		
0,600	14,97996	0,0527		
1/2	15,54806	0,0490	21,90	2,02
0,400	16,36810	0,0442		
1/3	17,08967	0,0405		
1/4	18,23278	0,0356		
1/5	19,07050	0,0325	26,25	2,17
1/6	19,70220	0,0305		
1/7	20,19310	0,0290		
1/8	20,58464	0,0279		
1/9	20,90385	0,0271		
1/10	21,16888	0,0264		
1/12	21,58327	0,0254		
1/15	22,01891	0,0244		
1/20	22,47701	0,0234		
1/50	23,36253	0,0217		
0,00	24,00000	**0,0205**	32,1	2,38

were retrieved by interpolation of data listed by Shah and London [9] (in their Table 47) and used to calculate n according to equation (16).

Parallel plates

In Table 2, the limiting case of a rectangular shaped duct with an infinitely large aspect ratio is considered to be representative of parallel plates. In Table 3, a rectangular duct with an aspect ratio of zero resembles parallel plates. In both these cases, the hydrodynamic entrance length for parallel plates is given by

$$L_c^+ = 0{,}0205. \quad (21)$$

For parallel plates, using an n-value of 2,38 in equation (14), the following expression is obtained for the apparent friction factor:

$$f_{app}Re = 24\,[1 + (0{,}0205/x^+)^{1,19}]^{0,42}. \quad (22)$$

Values obtained using the present definition of entrance length deviate most severely from those of other definitions for the case of parallel plates, as is evident from Table 1. A preliminary finite difference solution to the full Navier-Stokes equations for this case was conducted for a Reynolds number of 200 and a 21 × 21 array of grid points. The 99% velocity criterion applied to these numerical results yields an entrance length of 0,012, close to the 0,011 listed by Fleming and Sparrow [3]. Conversely, the presently derived L_c^+ criterion of equation (21), when applied to the numerical data, conforms to a 99,77% development of the centreline velocity.

Numerical work shows that the entry length as defined by Mohanty and Asthana [2], which precedes the filled region, is much shorter than the entrance length. For the specific case mentioned above, where an entrance length of 0,0205 was obtained, the boundary layer seems to meet at a dimensionless axial distance, the entry length, of approximately 0,0035.

Isosceles triangular ducts

Isosceles triangular ducts with two sides of equal length are labelled according to the angle ϕ (in degrees) between these two sides. An equilateral duct is therefore an isosceles duct with $\phi = 60°$. Numerical data from Table 57 of Shah and London [9] was used to construct Table 4 in accordance with the present results.

Table 4: Isosceles triangular duct flows. $f\,Re$ – Shah and London [9] Table 57.

2ϕ	$f\,Re$	L_c^+	$(f_{app}Re)_c$	n
0°	12,000	0,0822		
30°	13,065	0,0693	19,06	1,84
60°	13,333	0,0666	19,20	1,90
90°	13,153	0,0684	18,69	1,97
120°	12,744	0,0729		
150°	12,226	0,0792		
180°	12,000	0,0822		

Discussion

McComas [11] presented a flow development analysis based on the approximation of an inviscid core of fluid in the entrance region. Coupling this with the 99% velocity criterion, McComas obtained a hydrodynamic entrance length of

$$L_{ic}^+ = \frac{(u_{max}/u_m)^2 - 1 - K(\infty)}{4f\,Re}. \quad (23)$$

Evaluation of the entrance length according to equation (23) is subject to prior knowledge of the numerical value of the incremental pressure drop number $K(\infty)$. It is known [9] that this expression underpredicts the entrance length considerably. This phenomenon can be seen in Table 1 where specific values for the McComas expression are compared with corresponding results of some other definitions cited above. Various sources, as indicated, were used to retrieve the numerical entries of Table 1.

The present results, and in particular equation (14), explicitly define the value of $f_{app}Re$ over the entire duct length. Comparison with tables and proposed empirical equations given by *inter alia* Shah and London [9] reveals that, despite its simplicity, equation (14) yields fairly accurate values of $f_{app}Re$ over all duct lengths for a wide variety of duct shapes. It also provides a sound basis from which deviations from fully developed flow solutions, resulting from solving of the full Navier-Stokes equations, can be studied.

The present analysis also provides two useful expressions, (11) and (14), for the friction factor-Reynolds number product. These expressions are applicable in both the developing and the fully developed regions of the duct.

Conclusions

A new way of defining the hydrodynamic developing length in the entrance section of a straight duct is proposed. It is physically sound, since it is based on known properties of the flow. The result is independent of the specific velocity profile at that point. The definiton is made in terms of the friction factor-Reynolds number product, $f\,Re$, of the fully developed region. The developing length (critical station, x_c^+) is usually obtainable analytically if the fully developed velocity profile is known.

Heat and momentum transfer phenomena are influenced by the local velocity profiles at the duct walls. Since the present definition of entrance length is a function of the apparent friction factor, which depends directly on flow conditions at the duct walls, it seems more appropriate to use this in engineering practice, rather than definitions based on the centreline velocity, which is only indirectly related to conditions prevailing at the walls.

If the proposed definition of hydrodynamic entrance length were accepted universally, correlation between results of different studies would be aided considerably as comparisons could be made on common ground. Since the entrance length is known beforehand, the specific velocity profile at that point may be computed immediately for comparison. The parameter n could also be used as a basis for comparison. Furthermore, if computational effort were focused on improving the accuracy of n for various common ducts, empirical equations would be obtained which describe the properties at any point within the entrance region. It would thus be possible to obtain more accurate information of the behaviour concerning flow in short ducts.

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