Cracks in plates under shear and compression

W. Endres*

(Received in Final Form December 1991)

Abstract

In gears, rolling elements like wheels or roller bearings or in blade attachments in turbomachinery, loading conditions exist, which in the contact area lead to high compressive stress and often superimposed shear stress. In welds, cracks or defects often remain in areas of residual stress. If high or low cycle fatigue shear is imposed, such cracks can propagate, but under compressive stress a defect may be less dangerous.

In this paper, a flat flaw or crack is represented by a slot of zero thickness and investigated under two-dimensional conditions. It is subject to shear and compression, which creates friction on the crack surface, reduces sliding and lowers the local stresses at the ends of the crack. Practical consequences are discussed.

Nomenclature

A, B, E complex constants
I, J complex functions
K_I stress intensity factor, opening mode due to tensile stress
K_{II} stress intensity factor, sliding mode due to shear stress
T surface shear force per unit area applied at the boundary
S surface tensile force per unit area applied at the boundary
P surface compressive force per unit area, $P = -S$
P_H hydrostatic compressive force per unit area.
a half length of slot or crack
r distance from crack tip, close field coordinate
x^*, y^* cartesian coordinates
x, y dimensionless cartesian coordinates $x = x^*/a$
z complex variable, $z = x + iy$
ζ complex variable, $\zeta = \xi + i\eta$
ξ, η elliptic coordinates
θ angle from x axis, close field coordinate
μ coefficient of friction
Subscripts:
x,y . . . refer to coordinates
1-4 after σ_y . . refer to load cases of Fig. 2

Introduction

In some cases material is subject simultaneously to shear and compressive stresses. Contact areas in roller bearings or between the railway wheel and the rail or in gears between the teeth are a few examples. Such conditions also prevail in blade attachments of turbomachinery. The stresses, shear and compression, or shear alone, can be of cyclic nature and can lead to fatigue cracks. Whilst under most conditions cracks originate on the surface, in some cases they can start underneath the surface. Several investigations have dealt with this problem [1,2]. In the case of rolling contact it has been found that the orthogonal shear stress acting in planes parallel to the surface can propagate cracks below the surface.

In welds, cracks and defects are most likely to occur in the root of the weld. In high quality welds that region is normally ground out, if it is accessible. If that is not done, the defects remain in a region which in many cases experiences residual compressive stress.

It has already been observed that the propagation of fatigue cracks under shear is impaired by compressive stresses acting on the crack [2]. Friction between the crack surfaces transmits shear forces and reduces the stresses at the crack ends.

In geology we find similar conditions: during an earthquake two plates in the earth's crust slide under shear against each other along a fault, which can be subject to compression at the same time. The sliding fault can be interpreted as a crack in a single elastic plate.

In these contexts stresses around a flat crack in an infinite plate are investigated as a two-dimensional problem. Classical methods of the theory of elasticity are used.

Stresses around a crack in an infinite plate

The solution to this problem is a special case of stresses around an elliptic hole. Complex potential functions introduced by N.I. Muskhelishvili [3] lead to solutions. Here the notations of S. Timoshenko and J. N. Goodier [4] are used.

A crack in the x-y plane is shown in Fig. 1a. It reaches from $z = x = -1$ to $z = x = 1$ along the x-axis. If we want to introduce a crack length 2a we can replace the dimensionless coordinates x,y by $x^* = x.a$ and $y^* = y.a$.

The complex plane $z = x + i.y$ can be related to a $\zeta = \xi + i\eta$ plane by

(1) $z = f(\zeta) = \cosh\zeta$

Curves of constant ξ, η appear as ellipses and hyperbolas shown in Fig. 1b.

The boundaries of the plate are

$\xi = 0$ at the crack, which can be interpreted as an ellipse or slot of zero thickness, and

$\xi = \infty$ at infinity.

The general solution for the stresses can be written as

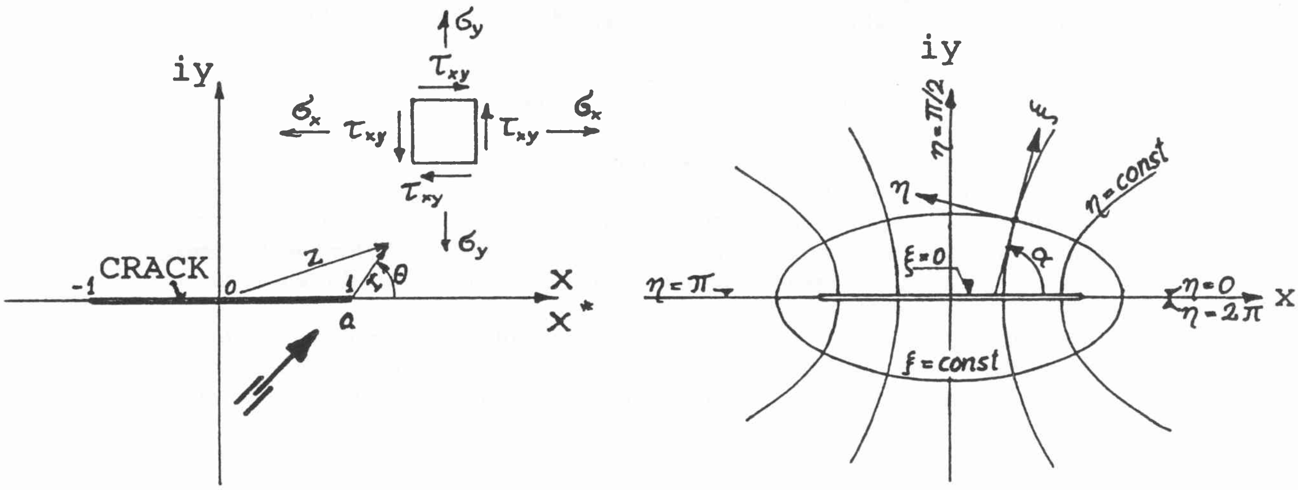
(2) $\sigma_x + \sigma_y = I = \mathrm{Re}(A + B.\coth\zeta) = \mathrm{Re}(A + B.z/\sqrt{z^2 - 1})$

(3) $\sigma_y - \sigma_x + 2i\tau_{xy} = J =$

$= -1/(2\sinh^3\zeta).[B\cosh\bar{\zeta}'] + (C + 2E)\cosh\zeta] + 2D + 2E.\coth\zeta$

$= -1/(2(z^2 - 1)^{\frac{3}{2}}).[B\,\bar{z} + (c + 2E)z] + 2D + 2E.z/\sqrt{z^2 - 1}$

* Dept of Mechanical Engineering, University of Stellenbosch



(a) Complex z = x + iy coordinate with crack extending from z = −1 (−1,0) to z = 1 (1,0)

(b) Lines of constant elliptic coordinates ξ, η shown in the x-y plane

Figure 1 Crack in infinite plate. The arrow in (a) refers to Fig. 7.

Re () means the real part, Im() the imaginary part. $\bar{\zeta}$ is the conjugate of ζ and $A = A_1 + iA_2$, B . . . E are complex constants.

The stresses are

(2a) $\sigma_y = \frac{1}{2}\,[I + Re(J)]$; $\sigma_x = \frac{1}{2}\,[I - Re(J)]$

(3a) $\tau_{xy} = \frac{1}{2}\,Im(J)$

Load cases

Let us first consider the following four load cases:

1. Tension $S > O$ across the crack applied at the boundary (at infinity) of the plate.
2. Bi-axial tension applied at the boundary of the plate.
3. Shear applied at the boundary of the plate.
4. Shear applied at the surface of the crack.

The crack opens under tension S. Under shear T the crack surfaces deform and slide on each other, but do not separate. If compressive stress $P = -S$ is applied, the crack is closed and P is transmitted across it. If shear T at infinity is added, a part of T is also transmitted through the crack surfaces as friction μP (μ being Coulomb's coefficient of friction). The rest $T - \mu P$ leads to sliding as in case 3.

The boundary forces S, T, P. (which have the dimension of stresses) for these cases are in the following set equal to unity, the stresses σ and τ become dimensionless. (To reverse this for values different from unity all stresses have to be multiplied by the value of S, T, P.)

The constants in equation (2) and (3) can be determined from the boundary conditions at the surfaces in the slot and at infinity and are summarized in the following table:

	Case 1	Case 2	Case 3	Case 4
A_1, A_2	−1,0	0,0	0,0	0,0
B_1, B_2	2,0	2,0	0,−2	0,2
C_1, C_2	−2,0	−2,0	0,0	0,0
D_1, D_2	0.5,0	0,0	0,0	0,1
E_1, E_2	0,0	0,0	0,1	0,−1

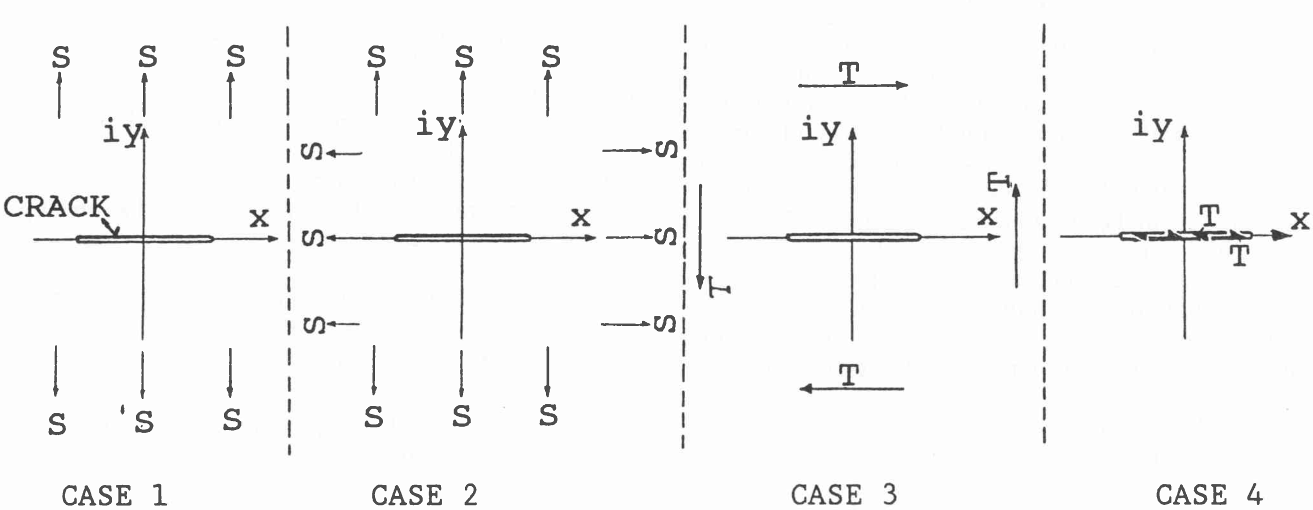


Figure 2 Load CASES 1-4

If case 3 and 4 are applied simultaneously (with equal T) uniform shear in the whole region results. The stresses relate as:

(4) $\sigma_{x4} = -\sigma_{x3}$; $\sigma_{y4} = -\sigma_{y3}$; $\tau_{xy4} = 1 - \tau_{xy3}$

The results for plane stress are represented in graphic form. Fig. 3 shows as an example the shear stress τ_{xy} for the Case 3. The vertical scale is given at the left. The region plotted extends from x = −2 to x = 2 and from y = 0 to y = 2. It shows one half of the plate and the crack. The stresses are symmetric with respect to the origin. They decline quickly with increasing distance from the crack tip and at the corners of the region the stresses are already close to the values reached at infinity. The disturbance caused by the crack is of very local nature.

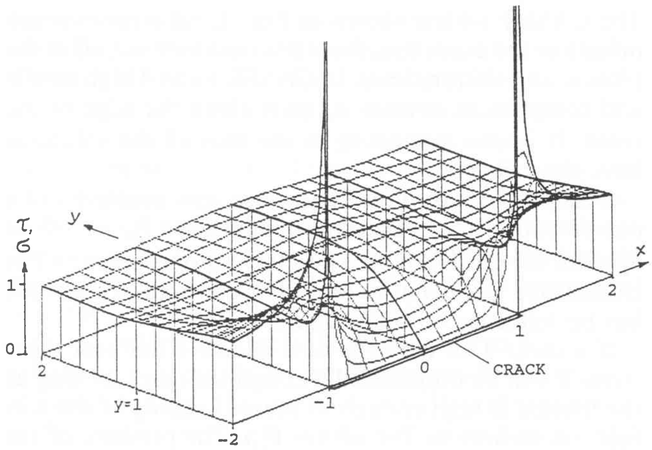


Figure 3 Shear stress τ_{xy} for the case 3 plotted over the x-y plane.

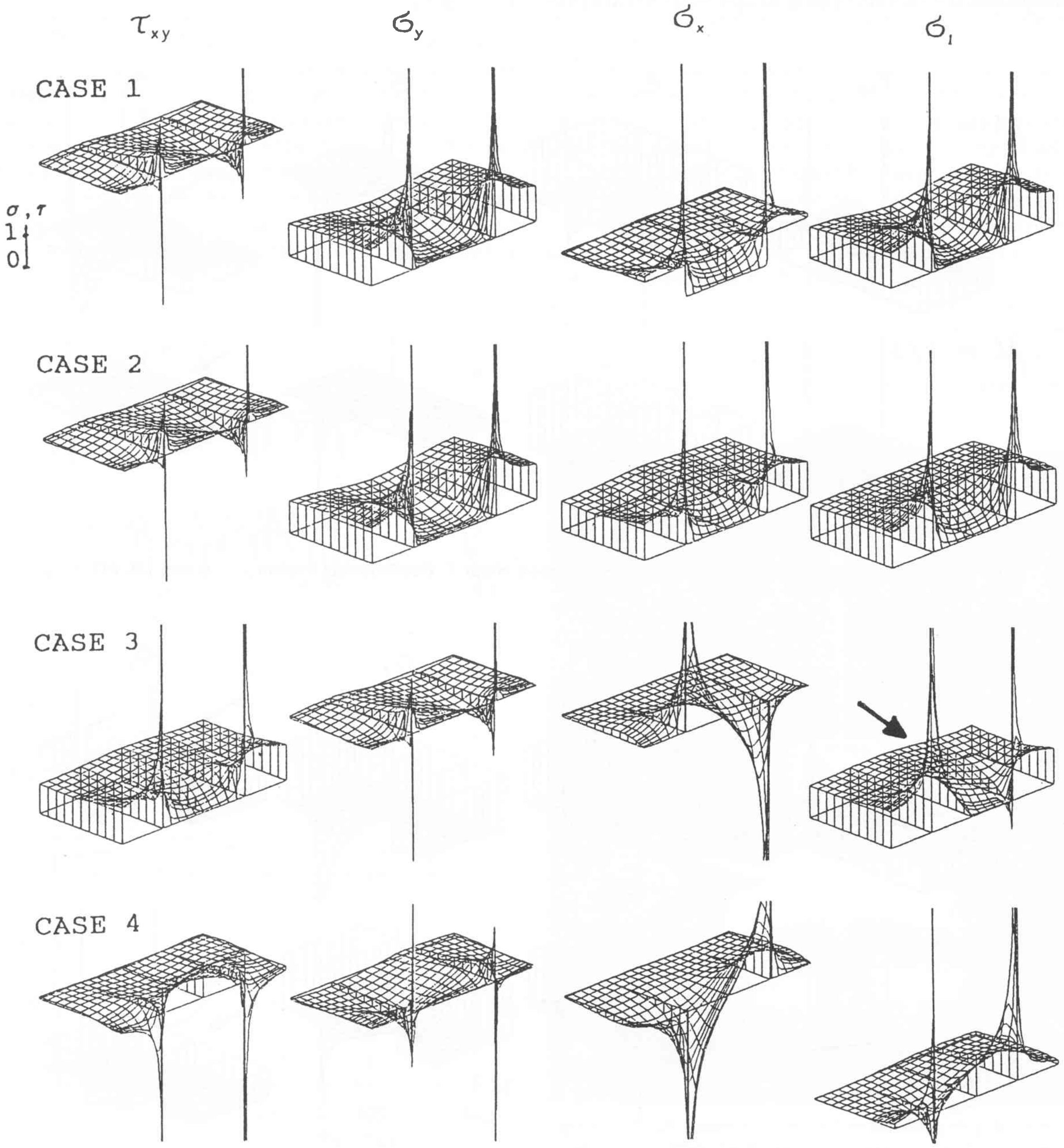


Figure 4 Stresses τ_{xy}, σ_y, σ_x, and first principal stress σ_1 for the CASES 1-4 plotted over the x-y plane.

The CASES 1-4 are shown in Fig. 4. All stresses reach infinity at the crack tips, the stress peaks are cut off in the plots at an arbitrary level. In CASES 3 and 4 high tensile and compressive stresses σ_x exist along the edge of the crack. It is also interesting to see how all the solutions have similarities.

If the load in case 1 (or 2) now becomes negative and a compression P (or hydrostatic compression $P_H = -S$) is applied, the crack is closed and the uniform pressure P is transmitted through the crack. The presence of the crack has no influence on the stresses.

If a shear T of case 3 is now added, a uniform shear stress T will be transmitted through the crack as long as the friction is high enough to prevent sliding of the surface, i.e. as long as $T < \mu P$ (or P_H). The presence of the crack again has no influence on the stresses.

Only if this friction limit is surpassed will sliding start and additional stresses according to case 4, which are proportional to $(T - \mu P)$ will be set up. Only in this case will the presence of the crack be felt and might crack growth start.

The superposition leads to

$$(5)\quad \begin{aligned} \sigma_y^* &= -P + T\,\sigma_{y3} + \mu P\,\sigma_{y4} &&= -P + (T - \mu P)\,\sigma_{y3} \\ \sigma_x^* &= T\,\sigma_{x3} + \mu P\,\sigma_{x4} &&= (T - \mu P)\,\sigma_{x3} \\ \tau_{xy}^* &= T\,\tau_{xy3} + \mu P\,\tau_{xy4} &&= \mu P + (T - \mu P)\,\tau_{xy3} \end{aligned}$$

(* indicates that these stresses now have the dimension of P,T) or in the case of hydrostatic pressure P_H

$$(6)\quad \begin{aligned} \sigma_y^* &= -P_H + T\,\sigma_{y3} + \mu P_H\,\sigma_{y4} &&= -P_H + (T - \mu\,P_H)\,\sigma_{y3} \\ \sigma_x^* &= -P_H + T\,\sigma_{x3} + \mu P_H\,\sigma_{x4} &&= -P_H + (T - \mu P_H)\,\sigma_{x3} \\ \tau_{xy}^* &= T\,\tau_{xy3} + \mu P_H\,\tau_{xy4} &&= \mu\,P_H + (T - \mu P_H) + \tau_{xy3} \end{aligned}$$

The equations on the right side show how the uniform state of stress is disturbed by the sliding deformation in the crack, as the shear T exceeds the friction force μ P. They are expressed by the stresses of case 3 multiplied by $(T - \mu\,P_H)$.

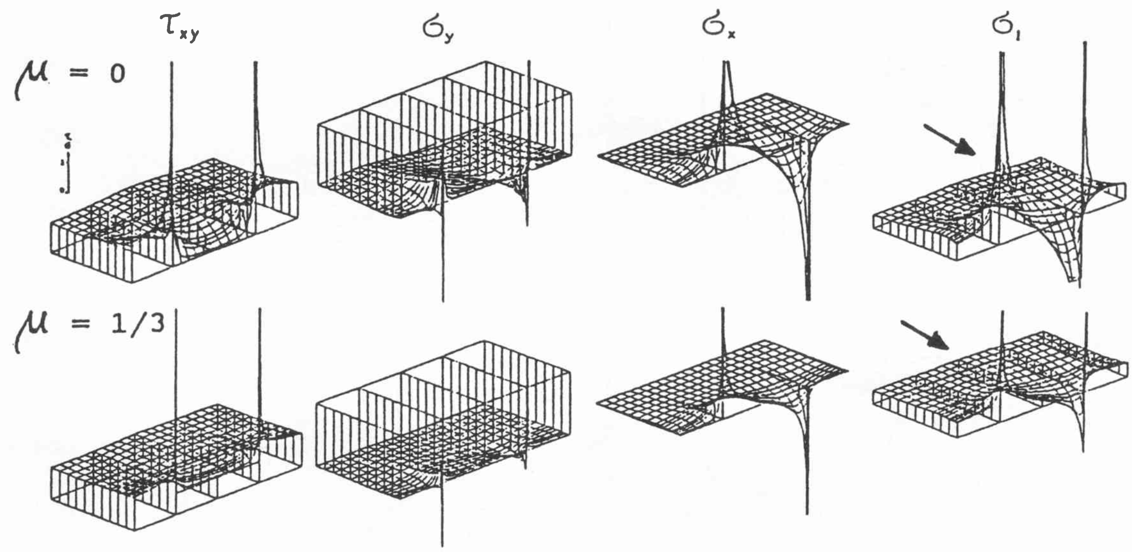


Figure 5 Stresses under compression P and superimposed shear T. Coefficient of friction μ = 0 and 1/3. P/T = 2.

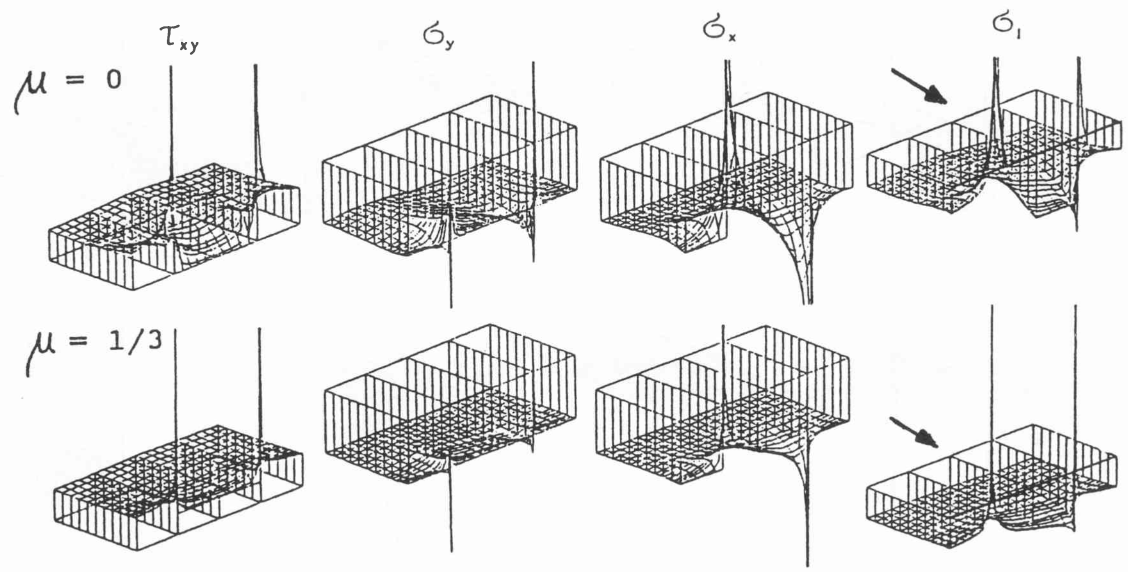


Figure 6 Stresses under hydrostatic compression P_H and superimposed shear T. Coefficient of friction μ = 0 and 1/3. P_H/T = 2.

Fig. 5 shows the stresses $\sigma = \sigma^*/T$ when the compression P is twice the shear T with a coefficient of friction $\mu = 1/3$ and, as a reference without friction. The first principal stress σ_1 has been added. Fig. 6 shows the same for the case of hydrostatic pressure P_H.

The volumes of the tensile peaks of the first principal stress, which are a measure of the elastic energy which can promote cracking, are strongly reduced as compression is applied and as the coefficient of friction increased. Hydrostatic pressure leaves even a smaller tensile peak than unilateral compression in the y-direction only. See arrows in Fig. 4 case 3, Fig. 5 and 6. Hydrostatic pressure should therefore reduce the danger of cracking more effectively.

Finally Fig. 7 shows a close-up of Fig. 5 at the crack tip at $x = 1$, seen in the direction of the arrow in Fig. 1a, i.e. under an angle of $\theta = -3/4\,\pi = -135°$.

The first principal stress σ_1 has a ridge of maxima in this direction. A crack due to this stress will start running from the crack tip at $x = 1$ perpendicularly to σ_1, i.e. under $\theta = -1/4\,\pi = -45°$.

One must always keep in mind that the problem is symmetric, i.e. both crack tips are equally prone to crack.

Fracture mechanics look only at the immediate neighbourhood of the cracks. Using local coordinates r, θ, see Fig. 1a, for small values of r the stresses at the crack tips transit to Westergaard's solution (see A. S. Kobayashi [5]). With the stress intensities

(7) $K_I = \sqrt{\pi a}\, S$ and $K_{II} = \sqrt{\pi a}\, T$ for our CASES 1 and 3, and keeping in mind that we have set a, S, T equal to unity, the stress (with dimensions: σ^*) become for CASE 1:

$$\sigma^*_{yI} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}.\sin\frac{3\theta}{2}\right]$$

(8) $$\sigma^*_{xI} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}.\sin\frac{3\theta}{2}\right].$$

$$\tau^*_{xyI} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}.\cos\frac{\theta}{2}.\cos\frac{3\theta}{2}$$

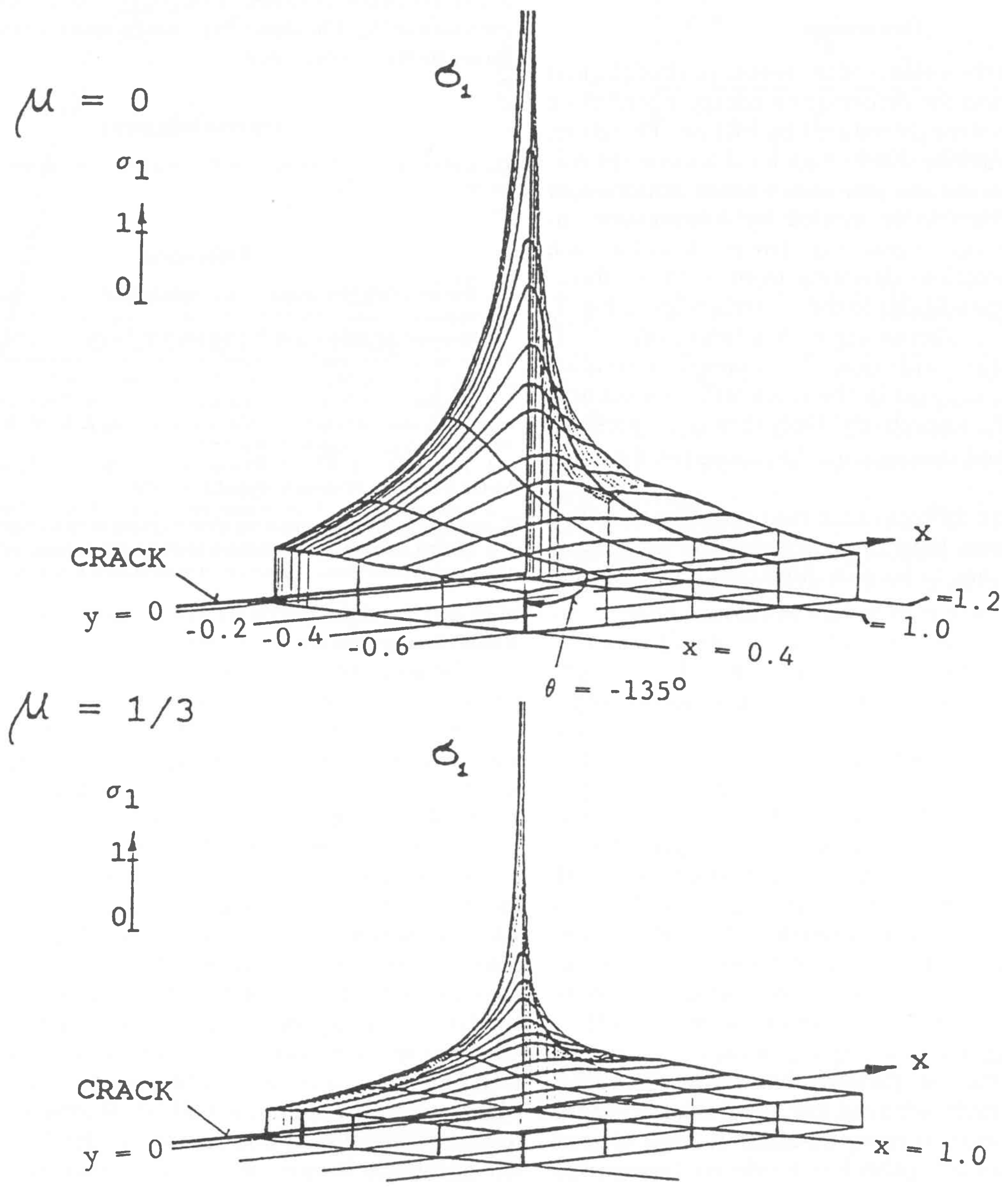


Figure 7 Close-up of $\sigma_1 = \sigma^*_1/T$ of Fig. 5 at the crack tip at x = 1, seen in the direction of the arrow in Fig. 1a.

And for CASE 3:

$$\sigma^*_{y3} = \frac{-K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\left[2 + \cos\frac{\theta}{2}.\cos\frac{3\theta}{2}\right]$$

$$(9)\ \sigma^*_{x3} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}.\sin\frac{\theta}{2}.\cos\frac{3\theta}{2}$$

$$\tau^*_{xy3} = \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}.\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

If compression and shear are superimposed the stress intensity factor of case 3 is reduced by the factor $(1 - \mu P/T)$ as can be found from equ. (5,6). The combined shear stress is:
$\tau_{xy}^* = \mu P + (T - \mu P)\ \tau_{xy3}$ and as the first term becomes small compared to the others close to the crack tip $(r \rightarrow 0)$:

(10) $\tau_{xy}^*/(T\ \tau_{xy3}) = (1 - \mu\ P/T) = K_{II}/K_{II3}$
For hydrostatic pressure P_H the same formula applies.

Discussion

It is obvious that the volume of the tensile peak of the first principal stress and the deformation energy stored close to the crack tip is strongly reduced by friction. Therefore, in cases where cracking due to high local tension (principal stress σ_1) is preponderant, like in brittle materials or under fatigue, the friction caused by compression increases the resistance to cracking. The crack will in such cases grow in directions deviating from its initial direction, roughly perpendicular to the σ_1 stress ridge in Fig. 7.

If the shear T is alternating with a total range $\triangle$ T, whilst P is constant with time (for example a residual stress in a weld), slippage in the crack will only occur if $\triangle T > 2\mu P$ or P_H respectively. Only then is the presence of the crack felt and stresses (equ. 5,6) caused by the crack occur.

If T and P have different time histories, like in rolling contact, the stresses have to be investigated stepwise.

However, one has to keep in mind that the coefficient of friction can vary widely, and can in the case of alternating load even change with the number of cycles. The value of this model is also limited by the fact that flaws or other defects are rarely flat and plane as assumed here. Cracks have significant asperities and roughness at a grain size level which would have a major effect on frictional interaction of a crack.

Conclusion

Flat flaws or cracks subject to shear have high stress peaks at the end of the cracks. If first compression with a resulting friction between the crack surfaces and then additional shear is applied, sliding in the crack is reduced, the peaks are narrower than without friction and the stress intensity K_{II} is reduced. Crack growth will start at a higher load level. This welcome increase in strength can be lost if the compressive stress is removed, for example, by stress relieving heat treatment of weld roots.

In contact problems damage normally originates at the surface despite the fact that the maximum shear stress in the Hertz pressure distribution or related cases lays below the surface [6]. The described mechanism certainly contributes to this experience.

Acknowledgment

The author wants to thank M. J. de V. Neethling for his support in plotting the results.

References

1. Kannel, J. W., Tevaarwerk, J. L., "Subsurface Stress Evalutions Under Rolling/Sliding Contacts," ASME Journal of Tribology, Vol. 106, 1984, pp. 96-106.
2. Sheppard, S., Barber, J. R., Comninou. M., "Short Subsurface Cracks Under Conditions of Stick and Slip Caused by a Moving Compressive Load," ASME Journal of Applied Mechanics, Vol. 52, 1985, pp. 811-817.
3. Muskhelishvili, N. I., "Some Basic Problems of the Theory of Elasticity", 4th augmented and corrected ed., 1954, translation by J. R. M. Radok, Erven P. Noordhoff, NV, Gronigen, Netherlands 1963.
4. Timoshenko, S. P. and Goodier, J. N., "Theory of Elasticity", 3rd ed., McGRAW-HILL KOGAGUSHA, LTD., 1970
5. Kobayashi, A. S., "Experimental Techniques in Fracture Mechanics", SESA monograph no. 1, THE IOWA STATE UNIVERSITY PRESS, First ed. 1973
6. Coutlakis, J. "Contact Stresses And Crack Mechanisms In Rolling/Sliding Contacts", Master thesis, University of Stellenbosch R.S.A. Nov. 1990.