A COMPLEX THREE-UNIT PARALLEL SYSTEM WITH PREPARATION TIME FOR REPAIR

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ABSTRACT

It is reasonable to expect that preparation time is needed to ready a repair facility before a repair can be carried out. A three-unit system with a `preparation time' for the repair facility is studied in this paper. The steady-state availability of such a system is obtained. The asymptotic confidence limits of the steady state availability are obtained numerically.

OPSOMMING

Redelikerwys kan verwag word dat ‘n herstelproses voorafgegaan word deur ‘n voorbereidingsproses voordat herstel ‘n aanvang kan neem. ‘n Sisteem wat bestaan uit drie eenhede waar ‘n sodanige voorbereidingsproses voor herstelwerk moet plaasvind word ondersoek. Die gestadigde stelselbeskikbaarheid met asimptotiese vertrouegrense word numeries blootgestel.

1. INTRODUCTION

Reliability theory is a very important branch of systems engineering and operations research and deals with methods of evaluating the various measures of performance of a system that may be subject to gradual deterioration. Any systems analysis, in order to be complete, must give due consideration to system reliability. Multiple unit systems have attracted the attention of many applied probabilists and reliability engineers for their applicability in their respective fields. Kistner and Subramanian [5] considered an *n*-unit warm standby redundant system with a single repair facility. In this case, the probability density function of the life time of the online unit was assumed to be arbitrary while all the other distributions are exponential; these results were later extended by Subramanian, Venkatakrishnan and Kistner [11]. Gupta and Bansal [3] studied the cost benefit analysis of a single server three unit redundant system with inspection, delayed replacement and two types of repair. A multiple component system in which n identical units connected in series are needed for the system to function, the units being supported by m spares and a single repair facility, Gupta and Bansal [3] have analyzed a cost function for a three unit standby system subject to random shocks and linearly increasing failure rates. The study of *n*-unit systems, even in the case of cold standbys, appears to be rather complicated. Sarma and Parvez [10] studied a three unit system in which all the distributions assumed are discrete. Muller [7] studied a three unit standby system when the life-times and repair time distributions are assumed to be arbitrary and obtained the expressions for reliability and availability. From the above literature, it is clear that all the models have the assumption that the repair facility is continuously available to attend to the repair of the failed units (Kistner and Subramanian [5], Krishnamoorthy et al [6], Bon and Paltanea [1], Frostig and Levikson [2] and Ke and Pearn [4]. But it is reasonable to expect that a preparation time might be needed to get the repair facility ready before the next repair could be taken up. If this preparation time is started only when a unit arrives for repair, it is easy to solve the problem, since the preparation time plus the actual repair time may be taken as the total repair time. But this preparation time usually starts immediately after each repair completion, so that the facility becomes available at the earliest. In this paper a three-unit standby redundant system is studied in which the preparation time has been introduced (Sarma [9]; Yadavalli et al [13], [14]). Asymptotic confidence limits for the steady state availability are also obtained. The numerical results are presented for the system measures in the last section.

2. SYSTEM DESCRIPTION

a. The system consists of three identical units connected in parallel. Either unit performs the system function satisfactorily.

b. There is only one repair facility. Each unit is new after repair.

c. At $t = 0$, all the units are new and the repair facility is available.

d. After each repair completion, the repair facility is not available for a random time which is called the `preparation time.

e. The life time, repair time and the preparation time are independent random variables and assumed to have an exponential distribution with parameters λ, μ, and γ respectively.

3. AVAILABILITY ANALYSIS

Consider the state of the system to be (i , j), where i is the number of failed units, and j is the state of the repair facility such that j = 0 represents that the repair facility is available and j = 1 represents that the repair facility is unavailable. The state transitions are presented in Table 1.

When n = 3, the possible transitions are presented in Table 2.

State		
From	To	Rate
(i , 0)	(i + 1, 0)	(n – i)λ, i = {0,1,…,n - 1}
(i , 0)	(i - 1, 0)	μ, i = {0,1,…,n}
(i , 1)	(i , 0)	γ, i = {0,1,…,n}
(i , 1)	(i + 1, 1)	(n- i)λ, i = {0,1,…,n - 1}

Table 1: State Transitions

State		
From	To	Rate
(0 , 0)	(1 , 0)	3λ
(0 , 1)	(0 , 0)	γ
(1,0)	(2,0)	2λ
(1,0)	(0,1)	μ
(0,0)	(1,0)	3λ
(1,1)	(1,0)	γ
(2,0)	(3,0)	λ
(2,0)	(1,1)	μ
(1,0)	(2,0)	2λ
(2,1)	(2,0)	γ
(3,0)	(2,1)	3μ
(2,0)	(3,0)	λ

Table 2: State transitions for n = 3

Figure 1 illustrates the possible states of the 3-unit system at any time and also the transition intensities. The balance equations are derived for the steady-state probabilities for the number of failed units in the system.

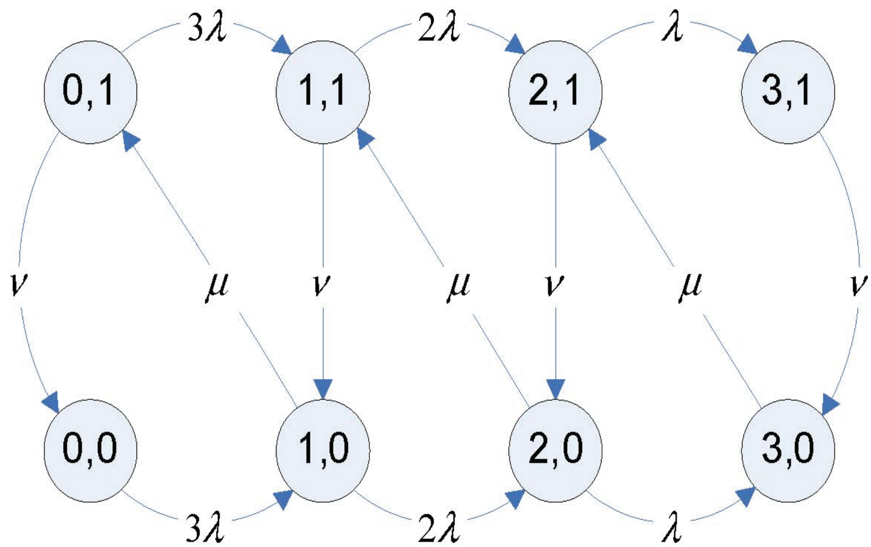


Figure 1: State transition diagram of a 3-unit system

Let

N (t) ≡ Number of failed units at time t.

R(t) ≡ State of the repair facility at time t.

Then {N(t) , R(t)} is a continuous time Markov process with the state space

S = {(i , j); i = 1, 2, 3; j = 0, 1}.

Let

$p_{ij}(t)$ = P[N(t) = i, R(t) = j]

$$p'_{00}(t) = -3\lambda p_{00}(t) + \gamma p_{01}(t) \qquad (1)$$
$$p'_{10}(t) = -(2\lambda + \mu) p_{10}(t) + 3\lambda p_{00}(t) + \gamma p_{11}(t) \qquad (2)$$
$$p'_{20}(t) = -(\lambda + \mu) p_{20}(t) + 2\lambda p_{10}(t) + \gamma p_{21}(t) \qquad (3)$$
$$p'_{30}(t) = -\mu p_{30}(t) + \lambda p_{20}(t) + \gamma p_{31}(t) \qquad (4)$$
$$p'_{01}(t) = -(3\lambda + \gamma) p_{01}(t) + \mu p_{10}(t) \qquad (5)$$
$$p'_{11}(t) = -(2\lambda + \gamma) p_{11}(t) + 3\lambda p_{01}(t) + \mu p_{20}(t) \qquad (6)$$
$$p'_{21}(t) = -(\lambda + \gamma) p_{21}(t) + 2\lambda p_{11}(t) + \mu p_{30}(t) \qquad (7)$$
$$p'_{31}(t) = -\gamma p_{31}(t) + \lambda p_{21}(t) \qquad (8)$$

In the steady-state

$$p_{ij} = \lim_{t \to \infty} P[N(t) = i, R(t) = j] \qquad (9)$$

Using (9), the steady-state equations for p_{ij} can be obtained as follows.

$$3\lambda p_{00} = \gamma p_{01} \qquad (10)$$
$$(2\lambda + \mu) p_{10} = 3\lambda p_{00} + \gamma p_{11} \qquad (11)$$
$$(\lambda + \mu) p_{20} = 2\lambda p_{10} + \gamma p_{21} \qquad (12)$$
$$\mu p_{30} = \lambda p_{20} + \gamma p_{31} \qquad (13)$$
$$(3\lambda + \gamma) p_{01} = \mu p_{10} \qquad (14)$$
$$(2\lambda + \gamma) p_{11} = 3\lambda p_{01} + \mu p_{20} \qquad (15)$$
$$(\lambda + \gamma) p_{21} = 2\lambda p_{11} + \mu p_{30} \qquad (16)$$
$$\gamma p_{31} = \lambda p_{21} \qquad (17)$$

Since the system is operable in states (1,0), (0,0), (2,0), (0,1), (1,1), and (2,1), the steady-state availability of the system is given by

$$A_{\infty} = \sum_{n-0}^{2} (p_{no} + p_{n1}) \qquad (18)$$

4. ESTIMATES FOR STEADY-STATE PROBABILITIES AND SYSTEM PERFORMANCE MEASURES

Let $X_1, X_2, \ldots, X_n$ be a random sample of failure times for operating units with probability density function (pdf)

$$f_1(x) = \lambda e^{-\lambda x} \qquad x > 0;\ \lambda > 0.$$

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of repair times of the failed units with pdf

$$f_2(y) = \mu e^{-\mu y} \qquad y > 0;\ \mu > 0.$$

Let $Z_1, Z_2, \ldots, Z_n$ be a sample of preparation times of the repair facility with pdf

$$f_3(z) = \gamma e^{-\gamma z} \qquad z > 0;\ \gamma > 0.$$

Let $\overline{X}$, $\overline{Y}$, $\overline{Z}$ be the sample means of the time to failure for the operating unit, the time to repair for the failed units, and the time to preparation for the repair facility respectively.

Then $E(\overline{X}) = \frac{1}{\lambda}$, $E(\overline{Y}) = \frac{1}{\mu}$ and $E(\overline{Z}) = \frac{1}{\gamma}$. It can be easily shown that $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are the maximum likelihood estimates of $\frac{1}{\lambda}$, $\frac{1}{\mu}$ and $\frac{1}{\gamma}$ respectively.

Furthermore, let $\hat{p}_{ij}$ be estimators of p_{ij}. The estimator of A_∞ can now be obtained through

$$\hat{A}_\infty = \sum_{n-0}^{2} (\hat{p}_{n0} + \hat{p}_{n1}) \qquad (19)$$

5. ASYMPTOTIC COFIDENCE LIMITS FOR THE AVAILABILITY

From the discussion in the previous section, $\hat{A}_\infty$ is a real-valued function in $\overline{X}$, $\overline{Y}$, $\overline{Z}$, which is also differentiable using the application of the multivariate central limit theorem due to Rao [8], it follows that

$\sqrt{n}\left[(X, Y, Z) - (\theta_1, \theta_2, \theta_3)\right]$ converges to $N_3(0,\Sigma)$ in the distribution as $n \to \infty$.

Where the dispersion matrix

$$\Sigma = \left[\sigma_{ij}^2\right]_{3x3}$$

is given by

$$\Sigma = diag\left[\theta_1^2, \theta_2^2, \theta_3^2\right],$$

using the results by Rao [8], it follows that

$$\sqrt{n}\left[\hat{A}_\infty - A_\infty\right] \xrightarrow{D} N_3(0, \sigma_1^2(\theta))$$

as n →∞ with

$$\sigma_1^2(\theta)=\sum_{i=1}^{3}\left[\frac{\partial A_\infty}{\partial \theta_i}\right]^2 \delta_{ii}$$

where

$$\theta=(\theta_1,\theta_2,\theta_3)$$

Let $\sigma_1^2(\hat{\theta})$ be the estimator for $\sigma_1^2(\theta)$ which is obtained by replacing θ by a consistent estimator $\hat{\theta}=(\overline{\mathrm{X}},\overline{\mathrm{Y}},\overline{\mathrm{Z}})$. Since $\sigma_1^2(\theta)$ is a continuous function of θ, $\sigma_1^2(\hat{\theta})$ is a consistent estimator of $\sigma_1^2(\theta)$ [12].

Therefore,

$$\sigma_1^2(\hat{\theta})\to\sigma_1^2(\theta)$$

as $n\to\infty$. Slutzky's theorem gives

$$\frac{\sqrt{n}\left[\hat{A}_\infty - A_\infty\right]}{\sigma_1^2(\hat{\theta})}\xrightarrow{D} N(0,1)$$

as $n\to\infty$, which leads to

$$P\left[-Z_{\frac{\alpha}{2}}\le\frac{\sqrt{n}\left[\hat{A}_\infty - A_\infty\right]}{\sigma_1^2(\theta)}\le Z_{\frac{\alpha}{2}}\right]=1-\alpha$$

where $Z_{\frac{\alpha}{2}}$ is determined from standard normal tables or statistical software packages. Hence, the asymptotic 100(1 - α)% confidence limits for A_∞ are given by

$$\hat{A}_\infty \pm Z_{\frac{\alpha}{2}}\frac{\delta(\hat{\theta})}{\sqrt{n}}.$$

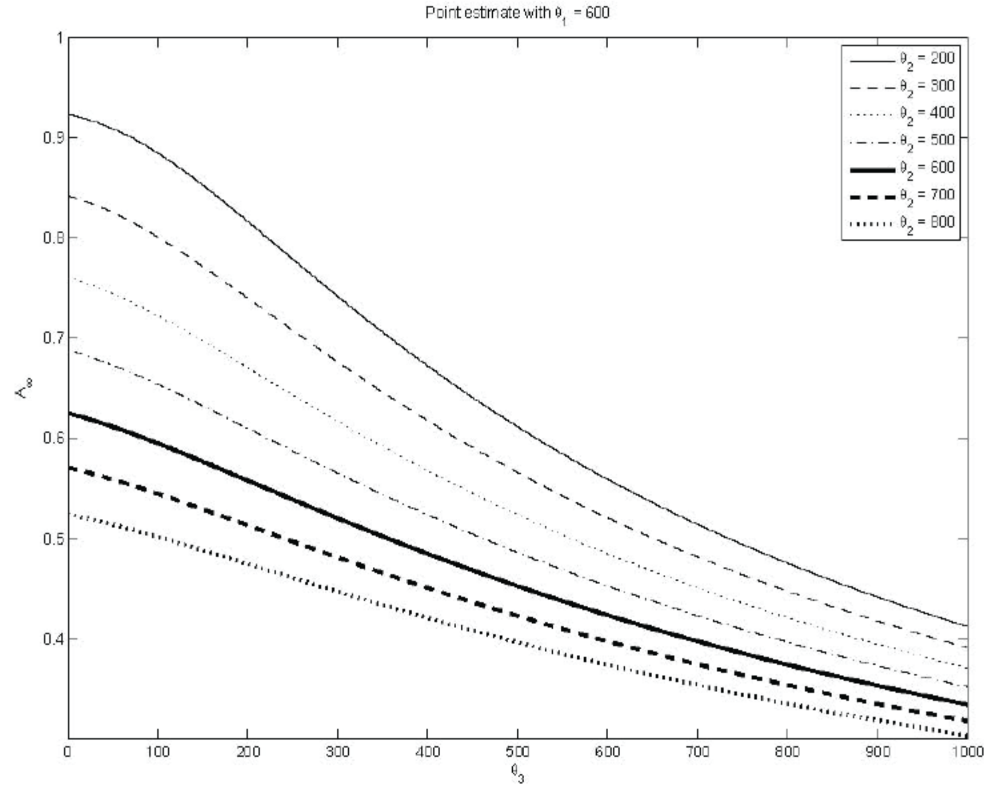
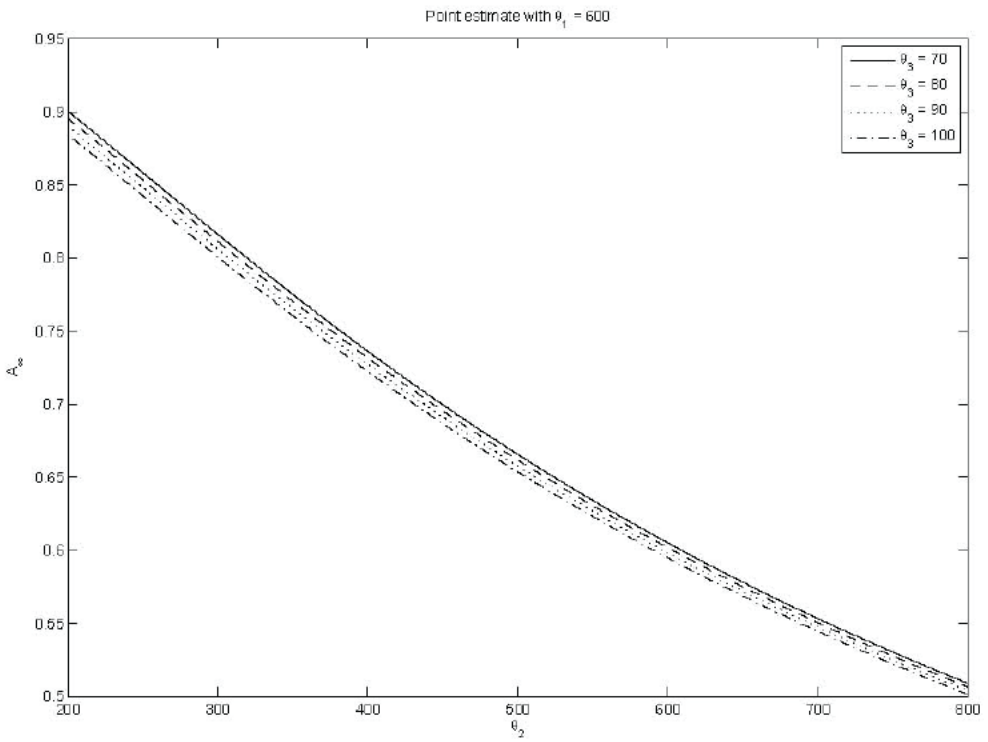


Figure 2: Steady state availability for changing repair time



n	θ_2	θ_3							
		70		80		90		100	
	200	0.8499	0.9509	0.8433	0.9476	0.8363	0.9441	0.8290	0.9402
	300	0.7415	0.8909	0.7357	0.8869	0.7297	0.8826	0.7234	0.8781
	400	0.6478	0.8245	0.6430	0.8203	0.6381	0.8160	0.6331	0.8115
100	500	0.5710	0.7602	0.5672	0.7563	0.5633	0.7522	0.5593	0.7481
	600	0.5087	0.7014	0.5056	0.6978	0.5024	0.6942	0.4992	0.6904
	700	0.4577	0.6489	0.4552	0.6457	0.4526	0.6424	0.4500	0.6391
	800	0.4155	0.6024	0.4134	0.5996	0.4113	0.5967	0.4091	0.5938
	200	0.8646	0.9361	0.8586	0.9324	0.8521	0.9283	0.8453	0.9239
	300	0.7634	0.8690	0.7578	0.8647	0.7521	0.8602	0.7461	0.8554
	400	0.6736	0.7986	0.6690	0.7944	0.6642	0.7900	0.6592	0.7854
200	500	0.5987	0.7325	0.5949	0.7286	0.5909	0.7245	0.5869	0.7204
	600	0.5369	0.6731	0.5337	0.6697	0.5305	0.6661	0.5272	0.6624
	700	0.4857	0.6209	0.4883	0.6178	0.4804	0.6146	0.4777	0.6114
	800	0.4429	0.5750	0.4407	0.5723	0.4384	0.5696	0.4362	0.5667
	200	0.8844	0.9163	0.8790	0.9120	0.8732	0.9072	0.8670	0.9022
	300	0.7926	0.8398	0.7874	0.8352	0.7819	0.8303	0.7763	0.8252
	400	0.7082	0.7661	0.7036	0.7597	0.6989	0.7552	0.6941	0.7505
1000	500	0.6357	0.6955	0.6318	0.6916	0.6279	0.6876	0.6238	0.6835
	600	0.5745	0.6355	0.5713	0.6321	0.5680	0.6286	0.5646	0.6250
	700	0.5230	0.5835	0.5203	0.5835	0.5203	0.5805	0.5175	0.5744
	800	0.4794	0.5385	0.4771	0.5359	0.4747	0.5333	0.4722	0.5306

Table 3: 95% Confidence interval for A_∞, θ_1 = 600

Figure 3: Steady state availability for changing preparation time

6. NUMERICAL ILLUSTRATION

In this section, numerical results of steady-state availability, A_∞ are given. Figure 2 illustrates the repair time (θ_3) vs A_∞ for fixed failure and preparation times respectively, while Figure 3 illustrates the preparation time (θ_2) vs A_∞ for fixed failure and repair time. From Figure 2 and3 it is clear that for fixed θ_1 as the repair time θ_2 increases, A_∞ decreases. Also, for a fixed θ_1, A_∞ decreases as the preparation time θ_3 increases. Tables 3 and 4 present the confidence limits (both at 95% and 99%) for different sample sizes. It is observed that, when n increases, the steady-state availability increases.

8. CONCLUSIONS

This paper develops the evaluation of availability using the difference differential equations for the state probabilities, for a three-unit complex system. The introduction of preparation time for the repair facility makes the system more complex.

The effects of various parameters on the system availability are numerically analyzed. Finally the sensitivity of system availability at specific values of parameters is examined. The numerical investigations indicate that as the preparation time increases the steady--state availability decreases this also holds for the repair time. The confidence limits for different parameters are obtained for different sample sizes. The results indicate that, when the sample size increases the availability increases, which is reasonable. For future study, non-Markovian models can be considered with the same and other assumptions

n	θ_2	θ_3							
		70		80		90		100	
100	200	0.8344	0.9664	0.8274	0.9636	0.8199	0.9606	0.8120	0.9573
	300	0.7186	0.9138	0.7126	0.9100	0.7063	0.9060	0.6998	0.9017
	400	0.6207	0.8516	0.6159	0.8475	0.6109	0.8432	0.6058	0.8388
	500	0.5420	0.7891	0.5382	0.7852	0.5634	0.7811	0.5304	0.7769
	600	0.4792	0.7309	0.4762	0.7272	0.4731	0.7235	0.4699	0.7197
	700	0.4284	0.6781	0.4260	0.6748	0.4235	0.6715	0.4210	0.6681
	800	0.3869	0.6310	0.3849	0.6281	0.3829	0.6251	0.3808	0.6220
200	200	0.8537	0.9470	0.8734	0.9436	0.8405	0.9400	0.8320	0.9360
	300	0.7472	0.8852	0.7415	0.8811	0.7355	0.8767	0.7293	0.8722
	400	0.6545	0.8177	0.6498	0.8136	0.6449	0.8092	0.6399	0.8047
	500	0.5782	0.7530	0.5744	0.7490	0.5705	0.7450	0.5665	0.7408
	600	0.5160	0.6940	0.5129	0.6905	0.5098	0.6868	0.5065	0.6831
	700	0.4650	0.6416	0.4624	0.6384	0.4598	0.6352	0.5272	0.6319
	800	0.4226	0.5953	0.4205	0.5925	0.4184	0.5896	0.4162	0.5867
1000	200	0.8795	0.9212	0.8739	0.9170	0.8680	0.9125	0.8616	0.9076
	300	0.7853	0.8471	0.7801	0.8425	0.7745	0.8377	0.7688	0.8327
	400	0.6996	0.7726	0.6951	0.7683	0.6903	0.7638	0.6855	0.7592
	500	0.6265	0.7047	0.6227	0.7008	0.6187	0.6968	0.6147	0.6927
	600	0.5652	0.6448	0.5620	0.6414	0.5587	0.6379	0.5553	0.6343
	700	0.5138	0.5928	0.5111	0.5898	0.5083	0.5867	0.5055	0.5836
	800	0.4704	0.5476	0.4681	0.5450	0.4657	0.5423	0.4633	0.5396

Table 4: 99% Confidence interval for A_∞, θ_1 = 600

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